

The Number of Representations of Positive Integers by Some Direct Sum Of Binary Quadratic Forms With Discriminant -103

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Abstract

A basis of a subspace of $S_4(\Gamma_0(103))$ has been found by means of generalized theta series associated to the direct sums of the quadratic forms $x_1^2 + x_1x_2 + 26x_2^2, 2x_1^2 \pm x_1x_2 + 13x_2^2, 4x_1^2 \pm 3x_1x_2 + 7x_2^2$.

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1. Introduction

Let's obtain a formula for the number of representation

$r(Q, n) = \#\{(x, y) \in \mathbb{Z}x\mathbb{Z}: x^2 + y^2 = n\}$ of a positive integer n by quadratic form $Q(x, y) = x^2 + y^2$. For this, we calculate the theta series

$$\begin{aligned} \Theta_Q(q) &= \sum_{n \in \mathbb{Z}} q^{n^2} \sum_{m \in \mathbb{Z}} q^{m^2} \\ &= \sum_{n, m \in \mathbb{Z}} q^{n^2 + m^2} \end{aligned} \quad (1.1)$$

$$= 1 + \sum_{n \in \mathbb{Z}} r(Q, n)q^n, q = e^{2\pi i\tau}, \text{Im}\tau > 0.$$

It is known that it is a modular form of weight 1 for the group

$\Gamma_0(4)$ with character χ_{-4} .

Here χ_{-4} is the Dirichlet character modulo 4 determined by

$$1 \rightarrow 1, 3 \rightarrow -1.$$

On the other hand, it is known that the space of modular forms of weight 1 with character χ_{-4} is one dimensional and generated by the Eisenstein series

$$\frac{1}{2}L(0, \chi_{-4}) + \sum_{n=1}^{\infty} \left(\sum_{d|n} \chi_{-4}(d) \right) q^n,$$

where the L-function

$$L(s, \chi_{-4}) = \sum_{n=1}^{\infty} \frac{\chi_{-4}(n)}{n^s}$$

defined for $\text{Res} > 0$ and can be analytically continued to the whole plane as holomorphic function. It can be shown that $L(0, \chi_{-4}) = \frac{1}{2}$.

Hence

$$\begin{aligned} \Theta_Q(q) &= 4 \cdot \left(\frac{1}{2} \cdot \frac{1}{2} + \sum_{n=1}^{\infty} \left(\sum_{d|n} \sum_{d \text{ is odd}} (-1)^{\frac{d-1}{2}} \right) q^n \right) \\ &= 1 + \sum_{n=1}^{\infty} 4 \left(\sum_{d|n} \sum_{d \text{ is odd}} (-1)^{\frac{d-1}{2}} \right) q^n, \end{aligned}$$

consequently, we obtain the simple formula

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$$r(Q, n) = 4 \left(\sum_{d|n, d \text{ is odd}} (-1)^{\frac{d-1}{2}} \right). \quad (1.2)$$

The immediate generalization to positive definite binary quadratic form Q of this formula is given by Dirichlet for discriminants Δ with class number $h(\Delta) = 1$ as follows:

$$r(Q, n) = w_{\Delta} \left(\sum_{d|n} \left(\frac{\Delta}{d} \right) \right), w_{\Delta} = \begin{cases} 6 & \text{for } \Delta = -3 \\ 4 & \text{for } \Delta = -4 \\ 2 & \text{for } \Delta < -4 \end{cases}$$

The determination of $r(Q, n)$ can be obtained by convolution products as in [1], [2] and [3]. But in this paper, we will obtain similar simple formulas as extensions of the work [4], [5] and [6] for direct sums of four positive definite quadratic forms of discriminant -103.

Since the class number of $\mathbb{Q}(\sqrt{-103})$ is 5, there exist 5 inequivalent classes of binary quadratic forms of discriminant -103 whose reduced forms, the ideals their norms and the corresponding complex numbers in the standard fundamental domain of $SL(2, \mathbb{Z})$ can be calculated easily as

$$F_1 = x_1^2 + x_1x_2 + 26x_2^2, \left[1, \frac{-1 + i\sqrt{103}}{2} \right] = \left[1, \frac{1 + i\sqrt{103}}{2} \right] = O_{-103},$$

$$N \left(\left[1, \frac{1 + i\sqrt{103}}{2} \right] \right) = 1, \frac{-1}{2} + \frac{i\sqrt{103}}{2}$$

$$\Phi_1 = 2x_1^2 + x_1x_2 + 13x_2^2, \left[2, \frac{-1 + i\sqrt{103}}{2} \right] = \left[2, 1 + \frac{1 + i\sqrt{103}}{2} \right],$$

$$N \left(\left[2, 1 + \frac{1 + i\sqrt{103}}{2} \right] \right) = 2, \frac{-1}{4} + \frac{i\sqrt{103}}{4}$$

$$\Psi_1 = 4x_1^2 + 3x_1x_2 + 7x_2^2, \left[4, \frac{-3 + i\sqrt{103}}{2} \right] = \left[4, -2 + \frac{1 + i\sqrt{103}}{2} \right],$$

$$N \left(\left[4, 2 + \frac{1 + i\sqrt{103}}{2} \right] \right) = 4, \frac{-3}{8} + \frac{i\sqrt{103}}{8}$$

$$\Psi'_1 = 4x_1^2 - 3x_1x_2 + 7x_2^2, \left[4, \frac{3 + i\sqrt{103}}{2} \right] = \left[4, 1 + \frac{1 + i\sqrt{103}}{2} \right]$$

$$N \left(\left[4, 1 + \frac{1 + i\sqrt{103}}{2} \right] \right) = 4, \frac{3}{8} + \frac{i\sqrt{103}}{8}$$

$$\Phi'_1 = M_1 = 2x_1^2 - x_1x_2 + 13x_2^2, \left[2, \frac{1 + i\sqrt{103}}{2} \right],$$

$$N \left(\left[2, \frac{1 + i\sqrt{103}}{2} \right] \right) = 2, \frac{1}{4} + \frac{i\sqrt{103}}{4}.$$

Here Φ'_1 is the inverse of Φ_1 , and they represent the same integers. Similarly, Ψ'_1 is the inverse of Ψ_1 and they represent the same integers. Therefore, the theta series of Φ_1 and Φ'_1 are the same with the theta series of Ψ_1 and Ψ'_1 respectively. F_1 is the identity element. The group of these quadratic forms is a group of order 5 and can be described in the following way.

Consider

$$\Phi_1 = 2x_1^2 + x_1x_2 + 13x_2^2, \gcd(2,1,2) = 1,$$

so by Dirichlet composition formula, there exists unique

$$b \bmod 2.2.2 = 8$$

such that

$$b \equiv 5 \bmod 4, b^2 \equiv -103 \bmod 4.2.2 = 16.$$

$$c = \frac{103 + 5^2}{16} = 8$$

So,

$$\begin{aligned} \varphi(x_1, x_2) &= 2 \cdot 2x_1^2 + 5x_1x_2 + 8x_2^2 \\ &= 4x_1^2 + 5x_1x_2 + 8x_2^2 \end{aligned}$$

which is equivalent to

$$4x_1^2 - 3x_1x_2 + 7x_2^2.$$

So,

$$\begin{aligned} \Phi_1^2 &= \Psi_1' \text{ and } \Phi_1^3 = \Phi_1 \Psi_1' \\ &= 8x_1^2 + 5x_1x_2 + 4x_2^2 \end{aligned}$$

which is equivalent to

$$4x_1^2 - 5x_1x_2 + 8x_2^2$$

, i.e.,

$$\Phi_1^3 = \Psi_1.$$

Therefore,

$\Phi_1, \Phi_1^2 = \Psi_1', \Phi_1^3 = \Psi_1, \Phi_1^4 = \Phi_1', \Phi_1^5 = F_1$.
Since 103 is prime, there is only one genus, i.e., principal genus.

Obviously, there are only two inequivalent cusps $i\infty$ and 0 for $\Gamma_0(103)$. We have the following important Theorem for the Eisenstein part of theta series associated to the quadratic form.

Theorem 1. Let Q be a positive definite form of $2k$ variables,

$$k = 4, 6, 8, \dots,$$

whose theta series Θ_Q is in $M_k(\Gamma_0(p))$, p prime, then the Eisenstein part of Θ_Q is

$$E(q; Q) = 1 + \sum_{n=1}^{\infty} (\alpha \sigma_{k-1}(n) q^n + \beta \sigma_{k-1}(n) q^{pn})$$

where

$$\begin{aligned} \alpha &= \frac{i^k p^{k/2} - i^k}{\rho_k p^k - 1}, \beta = \frac{1 p^k - i^k p^{k/2}}{\rho_k p^k - 1}, \rho_k \\ &= (-1)^{k/2} \frac{(k-1)!}{(2\pi)^k} \zeta(k). \end{aligned}$$

Proof. See [4].

We immediately obtain the following Corollary.

Corollary 2. Let Q be a positive definite form of 8 variables whose theta series Θ_Q is in

$$M_4(\Gamma_0(103)),$$

then the Eisenstein part of Θ_Q is

$$E(q; Q) = 1 + \sum_{n=1}^{\infty} (\alpha \sigma_3(n) q^n + \beta \sigma_3(n) q^{79n})$$

where

$$\rho_4 = \frac{3!}{(2\pi)^4} \zeta(4) = \frac{1}{240}$$

$$\begin{aligned} \alpha &= 240 \frac{103^2 - 1}{103^4 - 1} = 240 \frac{1}{103^2 + 1} \\ &= \frac{24}{1061}, \end{aligned}$$

$$\begin{aligned} \beta &= 240 \frac{103^4 - 103^2}{103^4 - 1} = 240 \frac{103^2}{103^2 + 1} \\ &= 103^2 \frac{24}{1061} \end{aligned}$$

and

$$\begin{aligned} E(q; F_4) &= E(q; F_3 \oplus \Phi_1) \\ &= E(q; F_2 \oplus \Phi_2) \\ &= E(q; F_1 \oplus \Phi_3) \\ &= E(q; \Phi_4) \end{aligned}$$

$$\begin{aligned} &= E(q; F_3 \oplus \Psi_1) = E(q; F_2 \oplus \Psi_2) \\ &= E(q; F_1 \oplus \Psi_3) \\ &= E(q; \Psi_4) \end{aligned}$$

$$\begin{aligned} &= E(q; \Phi_3 \oplus \Psi_1) = E(q; \Phi_2 \oplus \Psi_2) \\ &= E(q; \Phi_1 \oplus \Psi_3) = \dots \\ &= E(q; F_1 \oplus \Phi_1 \oplus \Psi_2) = \end{aligned}$$

$$\begin{aligned} &= 1 + \frac{24}{1061} \sum_{n=1}^{\infty} (q^n + 103^2 q^{103n}) \sigma_3(n) \\ &= 1 + \frac{24}{1061} \sum_{n=1}^{\infty} \sigma_3^*(n) q^n \end{aligned}$$

$$\begin{aligned}
 &= 1 + \frac{24}{1061} q + \frac{24 \cdot 9}{1061} q^2 + \frac{24 \cdot 28}{1061} q^3 \\
 &\quad + \frac{24 \cdot 73}{1061} q^4 + \frac{24 \cdot 126}{1061} q^5 \\
 &\quad + \frac{24 \cdot 252}{1061} q^6 \\
 &+ \frac{24 \cdot 344}{1061} q^7 + \frac{24 \cdot 585}{1061} q^8 + \frac{24 \cdot 757}{1061} q^9 \\
 &\quad + \frac{24 \cdot 1134}{1061} q^{10} \\
 &\quad + \frac{24 \cdot 1332}{1061} q^{11} \\
 &\quad + \frac{24 \cdot 2044}{1061} q^{12} + \frac{24 \cdot 2198}{1061} q^{13} \\
 &\quad + \frac{24 \cdot 3096}{1061} q^{14} \\
 &\quad + \frac{24 \cdot 3528}{1061} q^{15} \\
 &\quad + \frac{24 \cdot 4681}{1061} q^{16} \\
 &\quad + \frac{24 \cdot 4914}{1061} q^{17} + \frac{24 \cdot 6813}{1061} q^{18} \\
 &\quad + \frac{24 \cdot 6860}{1061} q^{19} \\
 &\quad + \frac{24 \cdot 9198}{1061} q^{20} \\
 &\quad + \frac{24 \cdot 9632}{1061} q^{21} \\
 &+ \frac{24 \cdot 11988}{1061} q^{22} + \frac{24 \cdot 12168}{1061} q^{23} \\
 &\quad + \frac{24 \cdot 16380}{1061} q^{24} \\
 &\quad + \frac{24 \cdot 15751}{1061} q^{25} + \dots
 \end{aligned}$$

where $\sigma_3^*(n) =$

$$\begin{cases} \sigma_3(n) & \text{if } n \geq 1 \text{ and } 103 \nmid n \\ \sigma_3(n) + 103^2 \sigma_3(n/103) & \text{if } 103 | n \end{cases}$$

1. The Positive Definite Forms

Now we will give some definitions, an important Theorem and evaluation of our quadratic forms.

Definition 1. Let $Q: \mathbb{Z}^{2k} \rightarrow \mathbb{Z}$ be a positive definite integer-valued form of $2k$ variables

$$Q = \sum_{1 \leq i \leq j \leq 2k}^{2k} b_{ij} x_i x_j, b_{ij} \in \mathbb{Z}$$

and the matrix A is defined by

$$a_{ii} = 2b_{ii}, a_{ji} = a_{ij} = b_{ij} \text{ for } i < j.$$

Let D be the determinant of the quadratic form

$$2Q = \sum_{i,j=1}^{2k} a_{ij} x_i x_j$$

, i.e., the determinant of the matrix A . Let A_{ij} be the cofactors of a_{ij} for $1 \leq i, j \leq 2k$. If

$\delta = \gcd\left(\frac{A_{ii}}{2}, A_{ij} \text{ for } 1 \leq i, j \leq 2k\right)$, then

$N := \frac{D}{\delta}$ is the smallest positive integer, called

the level of Q , for which

NA^{-1} is again an even integral matrix like A .

$\Delta = (-1)^k D$ is called the discriminant of the form Q .

Theorem 1. Let $Q: \mathbb{Z}^{2k} \rightarrow \mathbb{Z}$ be a positive definite integer-valued form of $2k$ variables of level N and discriminant Δ . Then 1-the theta function

$$\begin{aligned}
 \Theta_Q(q) &= \sum_{(n_1, n_2, \dots, n_k) \in \mathbb{Z} \times \mathbb{Z} \times \dots \times \mathbb{Z}} q^{Q(n_1, n_2, \dots, n_k)} \\
 &= 1 + \sum_{n=1}^{\infty} r(n; Q) q^n, q \\
 &= e^{2\pi iz}
 \end{aligned}$$

is a modular form on $\Gamma_0(N)$ of weight k and character χ_Δ , i.e., $\Theta_Q \in M_k(\Gamma_0(N), \chi_\Delta)$,

where

$$\chi_\Delta(d) := \left(\frac{\Delta}{d}\right), d$$

$\in (\mathbb{Z}$

$/N\mathbb{Z})^\times, \left(\frac{\Delta}{d}\right)$ is the Kronecker character.

2 –The homogeneous quadratic polynomials in $2k$ variables

$$\varphi_{ij} = x_i x_j - \frac{1}{2k} \frac{A_{ij}}{D} 2Q, \quad 1 \leq i, j \leq 2k \quad (1.1)$$

are spherical functions of second order with respect to Q .

3-The generalized theta series

$$\Theta_{Q, \varphi_{ij}}(q) = \sum_{n=1}^{\infty} \left(\sum_{Q=n} \varphi_{ij} \right) q^n \quad (1.2)$$

is a cusp form in $S_{k+2}(\Gamma_0(N), \chi_{\Delta})$.

4-If two quadratic forms Q_1, Q_2 have the same level N and the characters $\chi_1(d), \chi_2(d)$ respectively, then the direct sum $Q_1 \oplus Q_2$ of the quadratic forms has the same level N and the character $\chi_1(d) \cdot \chi_2(d)$ [4].

Let

$$\begin{aligned} F_2 &= F_1 \oplus F_1, \quad \Phi_2 = \Phi_1 \oplus \Phi_1, \quad \Psi_2 \\ &= \Psi_1 \oplus \Psi_1, F_1 \oplus \Phi_1, F_1 \\ &\oplus \Psi_1, \Phi_1 \oplus \Psi_1 \end{aligned}$$

be direct sums of the quadratic forms. In this paper, we will obtain the generalized theta series determined by the spherical functions associated to these quadratic forms. Let

$$\begin{aligned} F_k &= F_1 + \dots + F_1 (k \text{ times}), \quad \Phi_k \\ &= \Phi_1 + \dots \\ &+ \Phi_1 (k \text{ times}), \quad \Psi_k \\ &= \Psi_1 + \dots + \Psi_1 (k \text{ times}) \end{aligned}$$

are k direct sums of the quadratic forms.

By Theorem 1, $F_1, \Phi_1, \Phi'_1, \Psi_1, \Psi'_1$ are quadratic forms whose theta series are in

$$M_1 \left(\Gamma_0(103), \left(\frac{-103}{d} \right) \right).$$

Therefore, again by Theorem 1

$$\begin{aligned} F_4, \Phi_4, \Phi_3 \oplus \Phi'_1, \Phi_2 \oplus \Phi'_2, \Phi_1 \\ \oplus \Phi'_3, \Phi'_4, \Psi_4, \Psi_3 \oplus \Psi'_1, \Psi_2 \\ \oplus \Psi'_2, \Psi_1 \oplus \Psi'_3, \Psi'_4, \end{aligned}$$

$$\begin{aligned} F_3 \oplus \Phi_1, F_3 \oplus \Phi'_1, F_2 \oplus \Phi_2, F_2 \oplus \Phi_1 \\ \oplus \Phi'_1, F_2 \oplus \Phi'_2, F_1 \oplus \Phi_3, F_1 \\ \oplus \Phi_2 \oplus \Phi'_1, \end{aligned}$$

$$\begin{aligned} F_1 \oplus \Phi_1 \oplus \Phi'_2, F_1 \oplus \Phi'_3, F_3 \oplus \Psi_1, F_3 \\ \oplus \Psi'_1, F_2 \oplus \Psi_2, F_2 \oplus \Psi_1 \\ \oplus \Psi'_1, F_2 \oplus \Psi'_2, F_1 \oplus \Psi_3, \end{aligned}$$

$$\begin{aligned} F_1 \oplus \Psi_2 \oplus \Psi'_1, F_1 \oplus \Psi_1 \oplus \Psi'_2, F_1 \oplus \Psi'_3, \Phi_3 \\ \oplus \Psi_1, \Phi_2 \oplus \Phi'_1 \oplus \Psi_1, \Phi_1 \\ \oplus \Phi'_2 \oplus \Psi_1, \end{aligned}$$

$$\begin{aligned} \Phi'_3 \oplus \Psi_1, \Phi_3 \oplus \Psi'_1, \Phi_2 \oplus \Phi'_1 \oplus \Psi'_1, \Phi_1 \\ \oplus \Phi'_2 \oplus \Psi'_1, \Phi'_3 \oplus \Psi'_1, \Phi_2 \\ \oplus \Psi_2, \Phi'_2 \oplus \Psi_2, \end{aligned}$$

$$\begin{aligned} \Phi_2 \oplus \Psi'_2, \Phi'_2 \oplus \Psi'_2, \Phi_1 \oplus \Phi'_1 \oplus \Psi_2, \Phi_1 \\ \oplus \Phi'_1 \oplus \Psi_1 \oplus \Psi'_1, \Phi_1 \oplus \Phi'_1 \\ \oplus \Psi'_2, \Phi_2 \oplus \Psi_1 \oplus \Psi'_1, \end{aligned}$$

$$\begin{aligned} \Phi'_2 \oplus \Psi_1 \oplus \Psi'_1, \Psi_3 \oplus \Phi_1, \Psi_2 \oplus \Psi'_1 \\ \oplus \Phi_1, \Psi_1 \oplus \Psi'_2 \oplus \Phi_1, \Psi'_3 \\ \oplus \Phi_1, \Psi_3 \oplus \Phi'_1, \Psi_2 \oplus \Psi'_1 \\ \oplus \Phi'_1, \end{aligned}$$

$$\begin{aligned} \Psi_1 \oplus \Psi'_2 \oplus \Phi'_1, \text{ or } \Psi'_3 \oplus \Phi'_1, F_2 \oplus \Phi_1 \\ \oplus \Psi_1, F_2 \oplus \Phi'_1 \oplus \Psi_1, F_2 \\ \oplus \Phi_1 \oplus \Psi'_1, F_2 \oplus \Phi'_1 \oplus \Psi'_1, \end{aligned}$$

$$\begin{aligned} F_1 \oplus \Phi_2 \oplus \Psi_1, F_1 \oplus \Phi'_2 \oplus \Psi_1, F_1 \oplus \Phi_2 \\ \oplus \Psi'_1, F_1 \oplus \Phi'_2 \oplus \Psi'_1, F_1 \\ \oplus \Phi_1 \oplus \Phi'_1 \oplus \Psi_1, \end{aligned}$$

$$\begin{aligned} F_1 \oplus \Phi_1 \oplus \Phi'_1 \oplus \Psi'_1, F_1 \oplus \Phi_1 \oplus \Psi_2, F_1 \\ \oplus \Phi'_1 \oplus \Psi_2, F_1 \oplus \Phi_1 \\ \oplus \Psi'_2, F_1 \oplus \Phi'_1 \oplus \Psi'_2, \end{aligned}$$

$$\begin{aligned} F_1 \oplus \Phi_1 \oplus \Psi_1 \oplus \Psi'_1, F_1 \oplus \Phi'_1 \oplus \Psi_1 \oplus \Psi'_1, \\ \text{are quadratic forms whose theta series are in} \\ M_4(\Gamma_0(103)). \end{aligned}$$

In this paper, we will obtain the formulas of $r(n; Q)$ for these quadratic forms

Now let's look at the positive definite quadratic forms of discriminant -103:

1-For the quadratic form $F_1 = x_1^2 + x_1x_2 + 26x_2^2$,

$$2F_1 = 2x_1^2 + 2x_1x_2 + 52x_2^2 = (x_1, x_2) \begin{pmatrix} 2 & 1 \\ 1 & 52 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

the determinant of the matrix and a cofactor are

$$D = 103, A_{22} = 2.$$

So $\delta = 1, N = D = 103$ and the discriminant is

$$\Delta = (-1)^{2/2}103 = -103.$$

The character of F_1 is the Kronecker symbol

$$\chi(d) = \left(\frac{-103}{d}\right).$$

2-For the quadratic form $\Phi_1 = 2x_1^2 + x_1x_2 + 13x_2^2$,

$$2\Phi_1 = 4x_1^2 + 2x_1x_2 + 26x_2^2 = (x_1, x_2) \begin{pmatrix} 4 & 1 \\ 1 & 26 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

the determinant of the matrix and a cofactor are

$$D = 103, A_{21} = -1.$$

So $\delta = 1, N = D = 103$ and the discriminant is

$$\Delta = (-1)^{2/2}103 = -103.$$

The character of Φ_1 is the Kronecker symbol

$$\chi(d) = \left(\frac{-103}{d}\right).$$

3-For the quadratic form $\Psi_1 = 4x_1^2 + 3x_1x_2 + 7x_2^2$,

$$2\Psi_1 = 8x_1^2 + 6x_1x_2 + 14x_2^2 = (x_1, x_2) \begin{pmatrix} 8 & 3 \\ 3 & 14 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

the determinant of the matrix and some cofactors are

$$D = 103, A_{21} = -3, \frac{A_{11}}{2} = 7.$$

So $\delta = 1, N = D = 103$ and the discriminant is

$$\Delta = (-1)^{2/2}103 = -103.$$

The character of Ψ_1 is the Kronecker symbol

$$\chi(d) = \left(\frac{-103}{d}\right),$$

Consequently, $F_1, \Phi_1, \Phi'_1, \Psi_1, \Psi'_1$ are quadratic forms whose theta series are in

$$M_1\left(\Gamma_0(103), \left(\frac{-103}{d}\right)\right),$$

hence by Theorem 1,

$$F_2 = F_1 + \Phi_1, \Phi_2 = \Phi_1 + \Phi'_1, \Psi_2 = \Psi_1 + \Psi'_1, F_1 + \Phi_1, F_1 + \Psi_1, \Phi_1 + \Psi_1$$

are quadratic forms whose theta series are in

$$M_2(\Gamma_0(103)).$$

1. Spherical Functions

Here we will find all spherical functions such that the corresponding generalized theta series span all the generalized theta series of the form 2.4 induced by spherical functions of the form 2.3.

1-For quadratic form

$$2F_2 = 2x_1^2 + 2x_1x_2 + 52x_2^2 + 2x_3^2 + 2x_3x_4 + 52x_4^2$$

$$= (x_1, x_2, x_3, x_4) \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 52 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 52 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix},$$

the determinant and the matrix of cofactors are

$$D = 103^2, \begin{pmatrix} 5356 & -103 & 0 & 0 \\ -103 & 206 & 0 & 0 \\ 0 & 0 & 5356 & -103 \\ 0 & 0 & -103 & 206 \end{pmatrix}$$

By putting, $2k = 4, Q = F_2$, and appropriate i, j in Theorem 1, we get

$$\varphi_{11} = x_1^2 - \frac{15356}{4 \cdot 103^2} 2F_2 = x_1^2 - \frac{26}{103} F_2,$$

$$\varphi_{12} = x_1x_2 + \frac{1 \cdot 103}{4 \cdot 103^2} 2F_2 = x_1x_2 + \frac{1}{206} F_2,$$

which will be a spherical function of second order with respect to F_2 .

2-For quadratic form

$$2\Phi_2 = 4x_1^2 + 2x_1x_2 + 26x_2^2 + 4x_3^2 + 2x_3x_4 + 26x_4^2$$

$$= (x_1, x_2, x_3, x_4) \begin{pmatrix} 4 & 1 & & \\ 1 & 26 & & \\ & & 4 & 1 \\ & & 1 & 26 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix},$$

By putting $2k = 4, Q = \Phi_2$, and appropriate i, j in Theorem 1, we get

$$\varphi_{11} = x_1^2 - \frac{1 \cdot 2678}{4 \cdot 103^2} 2\Phi_2 = x_1^2 - \frac{13}{103} \Phi_2,$$

$$\begin{aligned} \varphi_{12} &= x_1x_2 + \frac{1 \cdot 103}{4 \cdot 103^2} 2\Phi_2 \\ &= x_1x_2 + \frac{1}{206} \Phi_2, \end{aligned}$$

$$\varphi_{22} = x_2^2 - \frac{1 \cdot 412}{4 \cdot 103^2} 2\Phi_2 = x_2^2 - \frac{2}{103} \Phi_2,$$

which will be spherical functions of second order with respect to Φ_2 .

3-For quadratic form

$$2(F_1 \oplus \Phi_1) = 2x_1^2 + 2x_1x_2 + 52x_2^2 + 4x_3^2 + 2x_3x_4 + 26x_4^2$$

$$= (x_1, x_2, x_3, x_4) \begin{pmatrix} 2 & 1 & & \\ 1 & 52 & & \\ & & 4 & 1 \\ & & 1 & 26 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix},$$

the determinant and the matrix of cofactors are

$$D = 103^2, \begin{pmatrix} 5356 & -103 & 0 & 0 \\ -103 & 206 & 0 & 0 \\ 0 & 0 & 2678 & -103 \\ 0 & 0 & -103 & 412 \end{pmatrix}$$

By putting $2k = 4, Q = F_1 \oplus \Phi_1$, and appropriate i, j in Theorem 1, we get

$$\begin{aligned} \varphi_{11} &= x_1^2 - \frac{152 \cdot 103}{4 \cdot 103^2} 2(F_1 \oplus \Phi_1) \\ &= x_1^2 - \frac{26}{103} (F_1 \oplus \Phi_1), \end{aligned}$$

$$\begin{aligned} \varphi_{12} &= x_1x_2 + \frac{1 \cdot 103}{4 \cdot 103^2} 2(F_1 \oplus \Phi_1) \\ &= x_1x_2 + \frac{1}{206} (F_1 \oplus \Phi_1), \end{aligned}$$

$$\begin{aligned} \varphi_{22} &= x_2^2 - \frac{1 \cdot 206}{4 \cdot 103^2} 2(F_1 \oplus \Phi_1) \\ &= x_2^2 - \frac{1}{103} (F_1 \oplus \Phi_1), \end{aligned}$$

$$\begin{aligned} \varphi_{33} &= x_3^2 - \frac{1 \cdot 26 \cdot 103}{4 \cdot 103^2} 2(F_1 \oplus \Phi_1) \\ &= x_3^2 - \frac{13}{103} (F_1 \oplus \Phi_1), \end{aligned}$$

$$\begin{aligned} \varphi_{34} &= x_3x_4 + \frac{1 \cdot 103}{4 \cdot 103^2} 2(F_1 \oplus \Phi_1) \\ &= x_3x_4 + \frac{1}{206} (F_1 \oplus \Phi_1) \end{aligned}$$

$$\begin{aligned} \varphi_{44} &= x_4^2 - \frac{1 \cdot 4 \cdot 103}{4 \cdot 103^2} 2(F_1 \oplus \Phi_1) \\ &= x_4^2 - \frac{2}{103} (F_1 \oplus \Phi_1), \end{aligned}$$

which will be spherical functions of second order with respect to $F_1 \oplus \Phi_1$.

4-For quadratic form

$$2\Psi_2 = 8x_1^2 + 6x_1x_2 + 14x_2^2 + 8x_3^2 + 6x_3x_4 + 14x_4^2$$

$$= (x_1, x_2, x_3, x_4) \begin{pmatrix} 8 & 3 & & \\ 3 & 14 & & \\ & & 8 & 3 \\ & & 3 & 14 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix},$$

the determinant and the matrix of cofactors are

D

$$= 103^2, \begin{pmatrix} 1442 & -309 & 0 & 0 \\ -309 & 824 & 0 & 0 \\ 0 & 0 & 1442 & -309 \\ 0 & 0 & -309 & 824 \end{pmatrix}$$

By putting $2k = 4$, $Q = \Psi_2$, and appropriate i, j in Theorem 1, we get

$$\varphi_{11} = x_1^2 - \frac{11442}{4 \cdot 103^2} 2\Psi_2 = x_1^2 - \frac{7}{103} \Psi_2,$$

$$\begin{aligned} \varphi_{12} &= x_1 x_2 + \frac{13 \cdot 103}{4 \cdot 103^2} 2\Psi_2 \\ &= x_1 x_2 + \frac{3}{206} \Psi_2, \end{aligned}$$

$$\varphi_{22} = x_2^2 - \frac{18 \cdot 103}{4 \cdot 103^2} 2\Psi_2 = x_2^2 - \frac{4}{103} \Psi_2$$

which will be spherical functions of second order with respect to Ψ_2 .

5-For quadratic form

$$2(F_1 \oplus \Psi_1) = 2x_1^2 + 2x_1x_2 + 52x_2^2 + 8x_3^2 + 6x_3x_4 + 14x_4^2$$

$$= (x_1, x_2, x_3, x_4) \begin{pmatrix} 2 & 1 & & \\ & 52 & & \\ & & 8 & 3 \\ & & 3 & 14 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix},$$

the determinant and the matrix of cofactors are

D

$$= 103^2, \begin{pmatrix} 5356 & -103 & 0 & 0 \\ -103 & 206 & 0 & 0 \\ 0 & 0 & 1442 & -309 \\ 0 & 0 & -309 & 824 \end{pmatrix}$$

By putting $2k = 4$, $Q = F_1 \oplus \Psi_1$, and appropriate i, j in Theorem 1, we get

$$\begin{aligned} \varphi_{11} &= x_1^2 - \frac{152 \cdot 103}{4 \cdot 103^2} 2(F_1 \oplus \Psi_1) \\ &= x_1^2 - \frac{26}{103} (F_1 \oplus \Psi_1), \end{aligned}$$

$$\begin{aligned} \varphi_{12} &= x_1 x_2 + \frac{1 \cdot 103}{4 \cdot 103^2} 2(F_1 \oplus \Psi_1) \\ &= x_1 x_2 + \frac{1}{206} (F_1 \oplus \Psi_1), \end{aligned}$$

$$\begin{aligned} \varphi_{22} &= x_2^2 - \frac{12 \cdot 103}{4 \cdot 103^2} 2(F_1 \oplus \Psi_1) \\ &= x_2^2 - \frac{1}{103} (F_1 \oplus \Psi_1), \end{aligned}$$

$$\begin{aligned} \varphi_{33} &= x_3^2 - \frac{114 \cdot 103}{4 \cdot 103^2} 2(F_1 \oplus \Psi_1) \\ &= x_3^2 - \frac{7}{103} (F_1 \oplus \Psi_1), \end{aligned}$$

$$\begin{aligned} \varphi_{34} &= x_3 x_4 + \frac{1 \cdot 103}{4 \cdot 103^2} 2(F_1 \oplus \Psi_1) \\ &= x_3 x_4 + \frac{1}{206} (F_1 \oplus \Psi_1), \end{aligned}$$

$$\begin{aligned} \varphi_{44} &= x_4^2 - \frac{18 \cdot 103}{4 \cdot 103^2} 2(F_1 \oplus \Psi_1) \\ &= x_4^2 - \frac{4}{103} (F_1 \oplus \Psi_1), \end{aligned}$$

which will be spherical functions of second order with respect to $F_1 \oplus \Psi_1$.

6-For quadratic form

$$2(\Phi_1 \oplus \Psi_1) = 4x_1^2 + 2x_1x_2 + 26x_2^2 + 8x_3^2 + 6x_3x_4 + 14x_4^2$$

$$= (x_1, x_2, x_3, x_4) \begin{pmatrix} 4 & 1 & & \\ & 26 & & \\ & & 8 & 3 \\ & & 3 & 14 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix},$$

the determinant and the matrix of cofactors are

$$D = 103^2, \begin{pmatrix} 2678 & -103 & 0 & 0 \\ -103 & 412 & 0 & 0 \\ 0 & 0 & 1442 & -309 \\ 0 & 0 & -309 & 824 \end{pmatrix}$$

By putting $2k = 4$, $Q = \Phi_1 \oplus \Psi_1$, and appropriate i, j in Theorem 1, we get

$$\begin{aligned}\varphi_{11} &= x_1^2 - \frac{1}{4} \frac{26 \cdot 103}{103^2} 2(\Phi_1 \oplus \Psi_1) \\ &= x_1^2 - \frac{13}{103} (\Phi_1 \oplus \Psi_1),\end{aligned}$$

$$\begin{aligned}\varphi_{12} &= x_1 x_2 + \frac{1}{4} \frac{103}{103^2} 2(\Phi_1 \oplus \Psi_1) \\ &= x_1 x_2 + \frac{1}{206} (\Phi_1 \oplus \Psi_1),\end{aligned}$$

$$\begin{aligned}\varphi_{22} &= x_2^2 - \frac{1}{4} \frac{26 \cdot 14}{103^2} 2(\Phi_1 \oplus \Psi_1) \\ &= x_2^2 - \frac{182}{103^2} (\Phi_1 \oplus \Psi_1),\end{aligned}$$

$$\begin{aligned}\varphi_{23} &= x_2 x_3 + \frac{14 \cdot 14}{4} \frac{103}{103^2} 2(\Phi_1 \oplus \Psi_1) \\ &= x_2 x_3 + \frac{28}{103^2} (\Phi_1 \oplus \Psi_1),\end{aligned}$$

$$\begin{aligned}\varphi_{33} &= x_3^2 - \frac{1}{4} \frac{14 \cdot 103}{103^2} 2(\Phi_1 \oplus \Psi_1) \\ &= x_3^2 - \frac{7}{103} (\Phi_1 \oplus \Psi_1),\end{aligned}$$

$$\begin{aligned}\varphi_{34} &= x_3 x_4 + \frac{1}{4} \frac{103}{103^2} 2(\Phi_1 \oplus \Psi_1) \\ &= x_3 x_4 + \frac{1}{206} (\Phi_1 \oplus \Psi_1),\end{aligned}$$

$$\begin{aligned}\varphi_{44} &= x_4^2 - \frac{1}{4} \frac{8 \cdot 103}{103^2} 2(\Phi_1 \oplus \Psi_1) \\ &= x_4^2 - \frac{4}{103} (\Phi_1 \oplus \Psi_1),\end{aligned}$$

which will be spherical functions of second order with respect to $\Phi_1 \oplus \Psi_1$.

Now we will determine a basis of the subspace of $S_4(\Gamma_0(103))$ spanned by the generalized theta series of the form 2.4 induced by spherical functions of the form 2.3. The dimension of $S_4(\Gamma_0(103))$ is 25[8].

Theorem 1. The following generalized 27 theta series

$$\begin{aligned}\Theta_{F_2, \varphi_{11}}(q) &= \frac{1}{103} \sum_{n=1}^{\infty} \sum_{F_2=n} (103x_1^2 \\ &\quad - 26F_2) q^n,\end{aligned} \quad (1.1)$$

$$\begin{aligned}\Theta_{F_2, \varphi_{12}}(q) &= \frac{1}{206} \sum_{n=1}^{\infty} \sum_{F_2=n} (206x_1 x_2 \\ &\quad + F_2) q^n,\end{aligned} \quad (1.2)$$

$$\begin{aligned}\Theta_{\Phi_2, \varphi_{11}}(q) &= \frac{1}{103} \sum_{n=1}^{\infty} \sum_{\Phi_2=n} (103x_1^2 \\ &\quad - 13\Phi_2) q^n\end{aligned} \quad (1.3)$$

$$\begin{aligned}\Theta_{\Phi_2, \varphi_{12}}(q) &= \frac{1}{2 \cdot 103} \sum_{n=1}^{\infty} \sum_{\Phi_2=n} (2 \\ &\quad \cdot 103x_1 x_2 + \Phi_2) q^n\end{aligned} \quad (0.1)$$

$$\begin{aligned}\Theta_{\Phi_2, \varphi_{22}}(q) &= \frac{1}{103} \sum_{n=1}^{\infty} \sum_{\Phi_2=n} (103x_2^2 \\ &\quad - 2\Phi_2) q^n,\end{aligned} \quad (0.2)$$

$$\begin{aligned}\Theta_{F_1 \oplus \Phi_1, \varphi_{11}}(q) &= \frac{1}{103} \sum_{n=1}^{\infty} \sum_{F_1 \oplus \Phi_1=n} (103x_1^2 \\ &\quad - 26F_1 \oplus \Phi_1) q^n,\end{aligned} \quad (0.3)$$

$$\begin{aligned}\Theta_{F_1 \oplus \Phi_1, \varphi_{12}}(q) &= \frac{1}{2 \cdot 103} \sum_{n=1}^{\infty} \sum_{F_1 \oplus \Phi_1=n} (2 \\ &\quad \cdot 103x_1 x_2 + F_1 \oplus \Phi_1) q^n,\end{aligned} \quad (0.4)$$

$$\begin{aligned} & \Theta_{F_1 \oplus \Phi_1, \varphi_{22}}(q) \\ &= \frac{1}{103} \sum_{n=1}^{\infty} \sum_{F_1 \oplus \Phi_1 = n} (103x_2^2 - F_1 \oplus \Phi_1) q^n, \end{aligned} \quad (0.5)$$

$$\begin{aligned} & \Theta_{F_1 \oplus \Phi_1, \varphi_{33}}(q) \\ &= \frac{1}{103} \sum_{n=1}^{\infty} \sum_{F_1 \oplus \Phi_1 = n} (103x_3^2 - 13F_1 \oplus \Phi_1) q^n, \end{aligned} \quad (0.6)$$

$$\begin{aligned} & \Theta_{F_1 \oplus \Phi_1, \varphi_{34}}(q) \\ &= \frac{1}{2 \cdot 103} \sum_{n=1}^{\infty} \sum_{F_1 \oplus \Phi_1 = n} (2 \cdot 103x_3x_4 + F_1 \oplus \Phi_1) q^n, \end{aligned} \quad (0.7)$$

$$\begin{aligned} & \Theta_{F_1 \oplus \Phi_1, \varphi_{44}}(q) \\ &= \frac{1}{103} \sum_{n=1}^{\infty} \sum_{F_1 \oplus \Phi_1 = n} (103x_4^2 - 2F_1 \oplus \Phi_1) q^n, \end{aligned} \quad (0.8)$$

$$\begin{aligned} & \Theta_{\Psi_2, \varphi_{11}}(q) \\ &= \frac{1}{103} \sum_{n=1}^{\infty} \sum_{\Psi_2 = n} (103x_1^2 - 7\Psi_2) q^n, \end{aligned} \quad (0.9)$$

$$\begin{aligned} & \Theta_{\Psi_2, \varphi_{12}}(q) \\ &= \frac{1}{2 \cdot 103} \sum_{n=1}^{\infty} \sum_{\Psi_2 = n} (2 \cdot 103x_1x_2 + 3\Psi_2) q^n, \end{aligned} \quad (0.10)$$

$$\begin{aligned} & \Theta_{\Psi_2, \varphi_{22}}(q) \\ &= \frac{1}{103} \sum_{n=1}^{\infty} \sum_{\Psi_2 = n} (103x_2^2 - 4\Psi_2) q^n, \end{aligned} \quad (0.11)$$

$$\begin{aligned} & \Theta_{F_1 \oplus \Psi_1, \varphi_{11}}(q) \\ &= \frac{1}{103} \sum_{n=1}^{\infty} \sum_{F_1 \oplus \Psi_1 = n} (103x_1^2 - 26F_1 \oplus \Psi_1) q^n, \end{aligned} \quad (0.12)$$

$$\begin{aligned} & \Theta_{F_1 \oplus \Psi_1, \varphi_{12}}(q) \\ &= \frac{1}{2 \cdot 103} \sum_{n=1}^{\infty} \sum_{F_1 \oplus \Psi_1 = n} (2 \cdot 103x_1x_2 + F_1 \oplus \Psi_1) q^n, \end{aligned} \quad (0.13)$$

$$\begin{aligned} & \Theta_{F_1 \oplus \Psi_1, \varphi_{22}}(q) \\ &= \frac{1}{103} \sum_{n=1}^{\infty} \sum_{F_1 \oplus \Psi_1 = n} (103x_2^2 - F_1 \oplus \Psi_1) q^n, \end{aligned} \quad (0.14)$$

$$\begin{aligned} & \Theta_{F_1 \oplus \Psi_1, \varphi_{33}}(q) \\ &= \frac{1}{103} \sum_{n=1}^{\infty} \sum_{F_1 \oplus \Psi_1 = n} (103x_3^2 - 7(F_1 \oplus \Psi_1)) q^n, \end{aligned} \quad (0.15)$$

$$\begin{aligned} & \Theta_{F_1 \oplus \Psi_1, \varphi_{34}}(q) \\ &= \frac{1}{2 \cdot 103} \sum_{n=1}^{\infty} \sum_{F_1 \oplus \Psi_1 = n} (2 \cdot 103x_3x_4 + (F_1 \oplus \Psi_1)) q^n, \end{aligned} \quad (0.16)$$

$$\begin{aligned} & \Theta_{F_1 \oplus \Psi_1, \varphi_{44}}(q) \\ &= \frac{1}{103} \sum_{n=1}^{\infty} \sum_{F_1 \oplus \Psi_1 = n} (103x_4^2 - 4(F_1 \oplus \Psi_1)) q^n, \end{aligned} \quad (0.17)$$

$$\begin{aligned} & \Theta_{\Phi_1 \oplus \Psi_1, \varphi_{11}}(q) \\ &= \frac{1}{103} \sum_{n=1}^{\infty} \sum_{\Phi_1 \oplus \Psi_1 = n} (103x_1^2 - 13(\Phi_1 \oplus \Psi_1)) q^n, \end{aligned} \quad (0.18)$$

$$\begin{aligned} & \Theta_{\Phi_1 \oplus \Psi_1, \varphi_{12}}(q) \\ &= \frac{1}{2 \cdot 103} \sum_{n=1}^{\infty} \sum_{\Phi_1 \oplus \Psi_1 = n} (2 \\ & \cdot 103x_1x_2 \\ & + (\Phi_1 \oplus \Psi_1))q^n, \end{aligned} \quad (0.19)$$

$$\begin{aligned} & \Theta_{\Phi_1 \oplus \Psi_1, \varphi_{22}}(q) \\ &= \frac{1}{103^2} \sum_{n=1}^{\infty} \sum_{\Phi_1 \oplus \Psi_1 = n} (103^2x_2^2 \\ & - 182(\Phi_1 \oplus \Psi_1))q^n, \end{aligned} \quad (0.20)$$

$$\begin{aligned} & \Theta_{\Phi_1 \oplus \Psi_1, \varphi_{23}}(q) \\ &= \frac{1}{103^2} \sum_{n=1}^{\infty} \sum_{\Phi_1 \oplus \Psi_1 = n} (103^2x_2x_3 \\ & + 28(\Phi_1 \oplus \Psi_1))q^n, \end{aligned} \quad (0.21)$$

$$\begin{aligned} & \Theta_{\Phi_1 \oplus \Psi_1, \varphi_{33}}(q) \\ &= \frac{1}{103} \sum_{n=1}^{\infty} \sum_{\Phi_1 \oplus \Psi_1 = n} (103x_3^2 \\ & - 7(\Phi_1 \oplus \Psi_1))q^n, \end{aligned} \quad (0.22)$$

$$\begin{aligned} & \Theta_{\Phi_1 \oplus \Psi_1, \varphi_{34}}(q) \\ &= \frac{1}{2 \cdot 103} \sum_{n=1}^{\infty} \sum_{\Phi_1 \oplus \Psi_1 = n} (2 \\ & \cdot 103x_3x_4 \\ & + (\Phi_1 \oplus \Psi_1))q^n, \end{aligned} \quad (0.23)$$

$$\begin{aligned} & \Theta_{\Phi_1 \oplus \Psi_1, \varphi_{44}}(q) \\ &= \frac{1}{103} \sum_{n=1}^{\infty} \sum_{\Phi_1 \oplus \Psi_1 = n} (103x_4^2 \\ & - 4(\Phi_1 \oplus \Psi_1))q^n \end{aligned} \quad (0.24)$$

has rank 20 in $S_4(\Gamma_0(103))$ and the set

$$\left\{ \Theta_{F_2, \varphi_{11}}, \Theta_{\Phi_2, \varphi_{11}}, \Theta_{\Phi_2, \varphi_{12}}, \Theta_{F_1 \oplus \Phi_1, \varphi_{11}}, \right. \\ \left. \Theta_{F_1 \oplus \Phi_1, \varphi_{12}}, \Theta_{F_1 \oplus \Phi_1, \varphi_{33}}, \Theta_{F_1 \oplus \Phi_1, \varphi_{34}}, \right.$$

$$\Theta_{\Psi_2, \varphi_{11}}, \Theta_{\Psi_2, \varphi_{12}}, \Theta_{F_1 \oplus \Psi_1, \varphi_{11}}, \Theta_{F_1 \oplus \Psi_1, \varphi_{12}}, \\ \Theta_{F_1 \oplus \Psi_1, \varphi_{33}}, \Theta_{F_1 \oplus \Psi_1, \varphi_{34}}, \Theta_{\Phi_1 \oplus \Psi_1, \varphi_{11}},$$

$$\left. \Theta_{\Phi_1 \oplus \Psi_1, \varphi_{12}}, \Theta_{\Phi_1 \oplus \Psi_1, \varphi_{22}}, \Theta_{\Phi_1 \oplus \Psi_1, \varphi_{23}}, \right\} \\ \Theta_{\Phi_1 \oplus \Psi_1, \varphi_{33}}, \Theta_{\Phi_1 \oplus \Psi_1, \varphi_{34}}, \Theta_{\Phi_1 \oplus \Psi_1, \varphi_{44}}$$

is a basis of the subspace of $S_4(\Gamma_0(103))$ spanned by the generalized theta series of the form 2.4 induced by spherical functions of the form 2.3.

Proof. The series are cusp forms because of Theorem 1, 2).

$$F_2 = x_1^2 + x_1x_2 + 26x_2^2 + x_3^2 + x_3x_4 \\ + 26x_4^2 = n$$

has the following solutions for

$$n = 1 \Rightarrow x_1 = \pm 1, x_2 = 0, x_3 = 0, x_4 \\ = 0 \text{ and } x_1 = 0, x_2 = 0, x_3 \\ = \pm 1, x_4 = 0,$$

$$n = 2 \Rightarrow x_1 = \pm 1, x_2 = 0, x_3 = \pm 1, x_4 = 0,$$

$$n = 3 \Rightarrow \text{it has no integral solutions,}$$

$$n = 4 \Rightarrow x_1 = \pm 2, x_2 = 0, x_3 = 0, x_4 \\ = 0 \text{ and } x_1 = 0, x_2 = 0, x_3 \\ = \pm 2, x_4 = 0,$$

$$n = 5 \Rightarrow x_1 = \pm 1, x_2 = 0, x_3 = \pm 2, x_4 \\ = 0 \text{ and } x_1 = \pm 2, x_2 = 0, x_3 \\ = \pm 1, x_4 = 0,$$

$$n = 6 \Rightarrow \text{it has no integral solutions,}$$

$$n = 7 \Rightarrow \text{it has no integral solutions,}$$

$$n = 8 \Rightarrow x_1 = \pm 2, x_2 = 0, x_3 = \pm 2, x_4 = 0,$$

$$n = 9 \Rightarrow x_1 = \pm 3, x_2 = 0, x_3 = 0, x_4 \\ = 0 \text{ and } x_1 = 0, x_2 = 0, x_3 \\ = \pm 3, x_4 = 0,$$

$$n = 10 \Rightarrow x_1 = \pm 3, x_2 = 0, x_3 = \pm 1, x_4 = 0 \text{ and } x_1 = \pm 1, x_2 = 0, x_3 = \pm 3, x_4 = 0,$$

$n = 11 \Rightarrow$ it has no integral solutions,

$n = 12 \Rightarrow$ it has no integral solutions,

$$n = 13 \Rightarrow x_1 = \pm 2, x_2 = 0, x_3 = \pm 3, x_4 = 0 \text{ and } x_1 = \pm 3, x_2 = 0, x_3 = \pm 2, x_4 = 0,$$

$n = 14 \Rightarrow$ it has no integral solutions,

$n = 15 \Rightarrow$ it has no integral solutions,

$$n = 16 \Rightarrow x_1 = \pm 4, x_2 = 0, x_3 = 0, x_4 = 0 \text{ and } x_1 = 0, x_2 = 0, x_3 = \pm 4, x_4 = 0,$$

$$n = 17 \Rightarrow x_1 = \pm 1, x_2 = 0, x_3 = \pm 4, x_4 = 0 \text{ and } x_1 = \pm 4, x_2 = 0, x_3 = \pm 1, x_4 = 0,$$

$$n = 18 \Rightarrow x_1 = \pm 3, x_2 = 0, x_3 = \pm 3, x_4 = 0,$$

$n = 19 \Rightarrow$ it has no integral solutions,

$$n = 20 \Rightarrow x_1 = \pm 4, x_2 = 0, x_3 = \pm 2, x_4 = 0, \text{ and } x_1 = \pm 2, x_2 = 0, x_3 = \pm 4, x_4 = 0,$$

$n = 21 \Rightarrow$ it has no integral solutions,

$n = 22 \Rightarrow$ it has no integral solutions,

$n = 23 \Rightarrow$ it has no integral solutions,

$n = 24 \Rightarrow$ it has no integral solutions,

$$n = 25 \Rightarrow x_1 = \pm 5, x_2 = 0, x_3 = 0, x_4 = 0 \text{ and } x_1 = 0, x_2 = 0, x_3 = \pm 5, x_4 = 0,$$

$$x_1 = \pm 4, x_2 = 0, x_3 = \pm 3, x_4 = 0 \text{ and } x_1 = \pm 3, x_2 = 0, x_3 = \pm 4, x_4 = 0.$$

So

$$\begin{aligned} & \Theta_{F_2, \varphi_{11}}(q) \\ &= \frac{1}{103} \sum_{n=1}^{\infty} \sum_{F_2=n} (103x_1^2 - 26F_2)q^n \quad (0.25) \\ &= \frac{1}{103} ((103 \cdot 2 - 26 \cdot 4)q \\ & \quad + (103 \cdot 4 - 26 \cdot 2 \cdot 4)q^2 \\ & \quad + (103 \cdot 4 \cdot 2 - 26 \cdot 4 \cdot 4)q^4 \\ & \quad + (103 \cdot 1 \cdot 4 + 103 \cdot 4 \cdot 4 - 26 \cdot 5 \cdot 8)q^5 \\ & \quad + (103 \cdot 4 \cdot 4 - 26 \cdot 8 \cdot 4)q^8 \\ & \quad + (103 \cdot 9 \cdot 2 - 26 \cdot 9 \cdot 4)q^9 \\ & \quad + (103 \cdot 9 \cdot 4 + 103 \cdot 1 \cdot 4 - 26 \cdot 10 \cdot 8)q^{10} \\ & \quad + (103 \cdot 4 \cdot 4 + 103 \cdot 9 \cdot 4 - 26 \cdot 13 \cdot 8)q^{13} \\ & \quad + (103 \cdot 16 \cdot 2 - 26 \cdot 16 \cdot 4)q^{16} \\ & \quad + (103 \cdot 1 \cdot 4 + 103 \cdot 16 \cdot 4 - 26 \cdot 17 \cdot 8)q^{17} \\ & \quad + (103 \cdot 9 \cdot 4 - 26 \cdot 18 \cdot 4)q^{18} \\ & \quad + (103 \cdot 16 \cdot 4 + 103 \cdot 4 \cdot 4 - 26 \cdot 20 \cdot 8)q^{20} \\ & \quad + (103 \cdot 25 \cdot 2 + 103 \cdot 16 \cdot 4 + 103 \cdot 9 \cdot 4 - 26 \cdot 25 \cdot 12)q^{25} + \dots) \\ &= \frac{1}{103} (102q + 204q^2 + 408q^4 + 1020q^5 \\ & \quad + 816q^8 + 918q^9 \\ & \quad + 2040q^{10} \end{aligned}$$

$$+2652q^{13} + 1632q^{16} + 3468q^{17} \\ + 1836q^{18} + 4080q^{20} \\ + 7650q^{25} \dots),$$

and

$$\Theta_{F_2, \varphi_{12}}(q) \\ = \frac{1}{206} \sum_{n=1}^{\infty} \sum_{F_2=n} (206x_1x_2 \\ + F_2)q^n \quad (0.26) \\ = \frac{1}{103} (2q + 4q^2 + 8q^4 + 20q^5 + 16q^8 \\ + 18q^9 + 40q^{10} + 52q^{13} \\ + 32q^{16}$$

$$+ 68q^{17} + 36q^{18} + 80q^{20} + 150q^{25} \dots).$$

Similarly, by solving the equations

$$\Phi_2 = 2x_1^2 + x_1x_2 + 13x_2^2 + 2x_3^2 + x_3x_4 \\ + 13x_4^2 = n$$

for

$$n = 1, 2, \dots, 25,$$

we get

$$\Theta_{\Phi_2, \varphi_{11}}(q) \\ = \frac{1}{103} \sum_{n=1}^{\infty} \sum_{\Phi_2=n} (103x_1^2 \\ - 13\Phi_2)q^n \quad (0.27) \\ = \frac{1}{103} (102q^2 + 204q^4 + 408q^8 \\ + 1020q^{10} - 676q^{13} \\ - 522q^{14} - 1148q^{15} \\ - 650q^{16} - 130q^{18} - 164q^{19} + 2040q^{20} \\ - 660q^{21} - 228q^{22} \\ - 372q^{23} \\ - 436q^{24} - 540q^{25} + \dots),$$

$$\Theta_{\Phi_2, \varphi_{12}}(q) \\ = \frac{1}{2 \cdot 103} \sum_{n=1}^{\infty} \sum_{\Phi_2=n} (2 \\ \cdot 103x_1x_2 + \Phi_2)q^n \quad (0.28)$$

$$= \frac{1}{103} (4q^2 + 8q^4 + 16q^8 + 40q^{10} \\ + 26q^{13} - 178q^{14} + 60q^{15} \\ - 78q^{16} + 520q^{18} - 374q^{19} + 80q^{20} \\ - 656q^{21} - 324q^{22} \\ + 458q^{23} \\ + 508q^{24} + 924q^{25} + \dots),$$

$$\Theta_{\Phi_2, \varphi_{22}}(q) \\ = \frac{1}{103} \sum_{n=1}^{\infty} \sum_{\Phi_2=n} (103x_2^2 \\ - 2\Phi_2)q^n, \quad (0.29)$$

$$= \frac{1}{103} (-16q^2 - 32q^4 - 64q^8 - 160q^{10} \\ + 102q^{13} + 94q^{14} + 172q^{15} \\ + 106q^{16}$$

$$- 20q^{18} + 54q^{19} - 320q^{20} + 152q^{21} \\ + 60q^{22} + 22q^{23} + 28q^{24} \\ + 12q^{25} + \dots).$$

Similarly, by solving the equations

$$F_1 \oplus \Phi_1 = x_1^2 + x_1x_2 + 26x_2^2 + 2x_3^2 + x_3x_4 \\ + 13x_4^2 = n$$

for

$$n = 1, 2, \dots, 25,$$

we get

$$\Theta_{F_1 \oplus \Phi_1, \varphi_{11}}(q) \\ = \frac{1}{103} \sum_{n=1}^{\infty} \sum_{F_1 \oplus \Phi_1=n} (103x_1^2 \\ - 26F_1 \oplus \Phi_1)q^n \quad (0.30) \\ = \frac{1}{103} (154q - 104q^2 + 100q^3 + 616q^4 \\ + 1024q^6 - 416q^8 + 862q^9 \\ + 2564q^{11} + 400q^{12}$$

$$-676q^{13} - 1772q^{14} - 1148q^{15} + 1632q^{16} \\ + 464q^{17} + 3560q^{18} \\ - 2552q^{19}$$

$$-2100q^{20} + 780q^{22} - 624q^{23} + 2012q^{24} \\ + 4958q^{25} + \dots),$$

$$\Theta_{F_1 \oplus \Phi_1, \varphi_{12}}(q) \\ = \frac{1}{2 \cdot 103} \sum_{n=1}^{\infty} \sum_{F_1 \oplus \Phi_1 = n} (2 \quad (0.31) \\ \cdot 103x_1x_2 + F_1 \oplus \Phi_1) q^n$$

$$= \frac{1}{103} (q + 2q^2 + 6q^3 + 4q^4 + 12q^6 \\ + 8q^8 + 27q^9 + 22q^{11} \\ + 24q^{12}$$

$$+ 13q^{13} + 42q^{14} + 30q^{15} + 32q^{16} \\ + 102q^{17} + 90q^{18} + 57q^{19}$$

$$+ 80q^{20} + 88q^{22} + 115q^{23} + 96q^{24} \\ + 75q^{25} + \dots),$$

$$\Theta_{F_1 \oplus \Phi_1, \varphi_{22}}(q) \\ = \frac{1}{103} \sum_{n=1}^{\infty} \sum_{F_1 \oplus \Phi_1 = n} (103x_2^2 \quad (0.32) \\ - F_1 \oplus \Phi_1) q^n$$

$$= \frac{1}{103} (-2q - 4q^2 - 12q^3 - 8q^4 - 24q^6 \\ - 16q^8 - 54q^9 - 44q^{11} \\ - 48q^{12}$$

$$- 26q^{13} - 84q^{14} - 60q^{15} - 64q^{16} \\ - 204q^{17} - 180q^{18} \\ - 114q^{19}$$

$$- 160q^{20} - 176q^{22} - 230q^{23} - 192q^{24} \\ - 150q^{25} + \dots),$$

$$\Theta_{F_1 \oplus \Phi_1, \varphi_{33}}(q) \\ = \frac{1}{103} \sum_{n=1}^{\infty} \sum_{F_1 \oplus \Phi_1 = n} (103x_3^2 \quad (0.33) \\ - 13F_1 \oplus \Phi_1) q^n$$

$$= \frac{1}{103} (-26q + 154q^2 + 256q^3 - 104q^4 \\ + 100q^6 + 616q^8 + 946q^9 \\ - 160q^{11} + 1024q^{12}$$

$$- 338q^{13} - 886q^{14} - 368q^{15} - 626q^{16} \\ - 592q^{17} + 338q^{18} \\ + 3050q^{19}$$

$$- 20q^{20} + 1420q^{22} - 106q^{23} + 800q^{24} \\ - 1538q^{25} + \dots),$$

$$\Theta_{F_1 \oplus \Phi_1, \varphi_{34}}(q) \\ = \frac{1}{2 \cdot 103} \sum_{n=1}^{\infty} \sum_{F_1 \oplus \Phi_1 = n} (2 \quad (0.1) \\ \cdot 103x_3x_4 + F_1 \oplus \Phi_1) q^n$$

$$= \frac{1}{103} (q + 2q^2 + 6q^3 + 4q^4 + 12q^6 \\ + 8q^8 + 27q^9 + 22q^{11} \\ + 24q^{12}$$

$$+ 13q^{13} - 164q^{14} - 382q^{15} + 238q^{16} \\ + 514q^{17} - 322q^{18} \\ - 355q^{19}$$

$$- 332q^{20} + 88q^{22} - 709q^{23} + 920q^{24} \\ + 487q^{25} + \dots),$$

and

$$\Theta_{F_1 \oplus \Phi_1, \varphi_{44}}(q) \\ = \frac{1}{103} \sum_{n=1}^{\infty} \sum_{F_1 \oplus \Phi_1 = n} (103x_4^2 \quad (0.2) \\ - 2F_1 \oplus \Phi_1) q^n$$

$$\begin{aligned}
&= \frac{1}{103} (-4q - 8q^2 - 24q^3 - 16q^4 - 48q^6 \\
&\quad - 32q^8 - 108q^9 - 88q^{11} \\
&\quad - 96q^{12} + 154q^{13} \\
&\quad + 450q^{14} + 292q^{15} + 78q^{16} + 416q^{17} \\
&\quad + 52q^{18} - 22q^{19} + 504q^{20} \\
&\quad + 60q^{22} + 570q^{23} + 28q^{24} + 112q^{25} \\
&\quad + \dots).
\end{aligned}$$

Similarly, by solving the equations

$$\Psi_2 = 4x_1^2 + 3x_1x_2 + 7x_2^2 + 4x_3^2 + 3x_3x_4 + 7x_4^2 = n$$

for

$$n = 1, 2, \dots, 25,$$

Proof. we get

$$\begin{aligned}
&\Theta_{\Psi_2, \varphi_{11}}(q) \\
&= \frac{1}{103} \sum_{n=1}^{\infty} \sum_{\Psi_2=n} (103x_1^2 - 7\Psi_2) q^n \quad (0.3)
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{103} (94q^4 - 196q^7 + 170q^8 - 204q^{11} \\
&\quad + 152q^{12} - 578q^{14}
\end{aligned}$$

$$\begin{aligned}
&\quad - 428q^{15} + 340q^{16} + 348q^{17} - 184q^{18} \\
&\quad + 940q^{20}
\end{aligned}$$

$$\begin{aligned}
&+ 120q^{21} - 408q^{22} + 360q^{23} + 1020q^{24} \\
&\quad + 660q^{25} + \dots),
\end{aligned}$$

$$\begin{aligned}
&\Theta_{\Psi_2, \varphi_{12}}(q) \\
&= \frac{1}{2 \cdot 103} \sum_{n=1}^{\infty} \sum_{\Psi_2=n} (2 \cdot 103x_1x_2 + 3\Psi_2) q^n \quad (0.4)
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{103} (24q^4 + 42q^7 - 110q^8 + 132q^{11} \\
&\quad - 268q^{12} + 374q^{14}
\end{aligned}$$

$$\begin{aligned}
&\quad - 232q^{15} - 220q^{16} - 310q^{17} + 628q^{18} \\
&\quad + 240q^{20}
\end{aligned}$$

$$\begin{aligned}
&+ 92q^{21} + 264q^{22} + 276q^{23} - 660q^{24} \\
&\quad - 936q^{25} + \dots),
\end{aligned}$$

$$\begin{aligned}
&\Theta_{\Psi_2, \varphi_{22}}(q) \\
&= \frac{1}{103} \sum_{n=1}^{\infty} \sum_{\Psi_2=n} (103x_2^2 - 4\Psi_2) q^n \quad (0.5)
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{103} (-64q^4 + 94q^7 - 50q^8 + 60q^{11} \\
&\quad + 28q^{12} + 170q^{14}
\end{aligned}$$

$$\begin{aligned}
&\quad 344q^{15} - 100q^{16} - 66q^{17} - 164q^{18} \\
&\quad - 640q^{20}
\end{aligned}$$

$$\begin{aligned}
&\quad - 108q^{21} + 120q^{22} - 324q^{23} - 300q^{24} \\
&\quad + 24q^{25} + \dots).
\end{aligned}$$

Similarly, by solving the equations

$$F_1 \oplus \Psi_1 = x_1^2 + x_1x_2 + 26x_2^2 + 4x_3^2 + 3x_3x_4 + 7x_4^2 = n$$

for

$$n = 1, 2, \dots, 25,$$

we get

$$\begin{aligned}
&\Theta_{F_1 \oplus \Psi_1, \varphi_{11}}(q) \\
&= \frac{1}{103} \sum_{n=1}^{\infty} \sum_{F_1 \oplus \Psi_1=n} (103x_1^2 - 26F_1 \oplus \Psi_1) q^n \quad (0.6)
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{103} (154q + 408q^4 - 108q^5 - 364q^7 \\
&\quad - 20q^8 + 862q^9 + 504q^{11} \\
&\quad + 400q^{12}
\end{aligned}$$

$$\begin{aligned}
&\quad + 2356q^{13} - 728q^{14} - 1148q^{15} + 3676q^{16} \\
&\quad - 300q^{17} - 1684q^{18} \\
&\quad + 4080q^{20}
\end{aligned}$$

$$\begin{aligned}
 & -536q^{21} + 5516q^{23} + 4096q^{24} \\
 & \quad + 4958q^{25} + \dots), \\
 & \Theta_{F_1 \oplus \Psi_1, \varphi_{12}}(q) \\
 & = \frac{1}{2 \cdot 103} \sum_{n=1}^{\infty} \sum_{F_1 \oplus \Psi_1 = n} (2 \quad (0.7) \\
 & \quad \cdot 103x_1x_2 + F_1 \oplus \Psi_1) q^n \\
 & = \frac{1}{103} (q + 8q^4 + 10q^5 + 7q^7 + 40q^8 \\
 & \quad + 27q^9 + 22q^{11} + 24q^{12} \\
 & \quad + 26q^{13} \\
 & + 14q^{14} + 30q^{15} + 64q^{16} + 85q^{17} + 72q^{18} \\
 & \quad + 80q^{20} + 42q^{21} + 92q^{23} \\
 & \quad + 48q^{24} + 75q^{25} + \dots),
 \end{aligned}$$

$$\begin{aligned}
 & \Theta_{F_1 \oplus \Psi_1, \varphi_{22}}(q) \\
 & = \frac{1}{103} \sum_{n=1}^{\infty} \sum_{F_1 \oplus \Psi_1 = n} (103x_3^2 \quad (0.8) \\
 & \quad - (F_1 \oplus \Psi_1)) \\
 & = \frac{1}{103} (-2q - 16q^4 - 20q^5 - 14q^7 \\
 & \quad - 80q^8 - 54q^9 - 44q^{11} \\
 & \quad - 48q^{12} \\
 & - 52q^{13} - 28q^{14} - 60q^{15} - 128q^{16} \\
 & \quad - 170q^{17} - 144q^{18} \\
 & \quad - 160q^{20} \\
 & - 84q^{21} - 184q^{23} - 96q^{24} - 150q^{25} \\
 & \quad + \dots),
 \end{aligned}$$

$$\begin{aligned}
 & \Theta_{F_1 \oplus \Psi_1, \varphi_{33}}(q) \\
 & = \frac{1}{103} \sum_{n=1}^{\infty} \sum_{F_1 \oplus \Psi_1 = n} (103x_3^2 \quad (0.9) \\
 & \quad - 7(F_1 \oplus \Psi_1))
 \end{aligned}$$

$$\begin{aligned}
 & = \frac{1}{103} (-14q + 94q^4 + 272q^5 - 98q^7 \\
 & \quad + 58q^8 + 34q^9 - 308q^{11} \\
 & \quad + 76q^{12} \\
 & + 48q^{13} + 10q^{14} - 8q^{15} - 72q^{16} \\
 & \quad + 1694q^{17} + 1052q^{18} \\
 & \quad + 940q^{20} \\
 & + 1060q^{21} - 876q^{23} - 260q^{24} + 598q^{25} \\
 & \quad + \dots),
 \end{aligned}$$

$$\begin{aligned}
 & \Theta_{F_1 \oplus \Psi_1, \varphi_{34}}(q) \\
 & = \frac{1}{2 \cdot 103} \sum_{n=1}^{\infty} \sum_{F_1 \oplus \Psi_1 = n} (2 \quad (0.10) \\
 & \quad \cdot 103x_3x_4 + (F_1 \oplus \Psi_1)) \\
 & = \frac{1}{103} (q + 8q^4 + 10q^5 + 7q^7 - 166q^8 \\
 & \quad - 385q^9 + 22q^{11} - 388q^{12} \\
 & \quad + 26q^{13} \\
 & + 220q^{14} + 442q^{15} + 64q^{16} - 739q^{17} \\
 & \quad - 340q^{18} + 80q^{20} \\
 & - 782q^{21} + 504q^{23} - 364q^{24} + 75q^{25} \\
 & \quad + \dots),
 \end{aligned}$$

and

$$\begin{aligned}
 & \Theta_{F_1 \oplus \Psi_1, \varphi_{44}}(q) \\
 & = \frac{1}{103} \sum_{n=1}^{\infty} \sum_{F_1 \oplus \Psi_1 = n} (103x_4^2 \quad (0.11) \\
 & \quad - 4(F_1 \oplus \Psi_1)) \\
 & = \frac{1}{103} (-8q - 64q^4 - 80q^5 + 150q^7 \\
 & \quad + 298q^8 + 196q^9 + 236q^{11} \\
 & \quad + 220q^{12} - 208q^{13} \\
 & + 94q^{14} + 172q^{15} - 100q^{16} - 62q^{17} \\
 & \quad + 248q^{18} - 640q^{20} \\
 & + 76q^{21} + 88q^{23} + 28q^{24} - 600q^{25} + \dots).
 \end{aligned}$$

Similarly, by solving the equations

$$\Phi_1 \oplus \Psi_1 = 2x_1^2 + x_1x_2 + 13x_2^2 + 4x_3^2 + 3x_3x_4 + 7x_4^2 = n$$

for

$$n = 1, 2, \dots, 25,$$

we

get

$$\begin{aligned} & \Theta_{\Phi_1 \oplus \Psi_1, \varphi_{11}}(q) \\ &= \frac{1}{103} \sum_{n=1}^{\infty} \sum_{\Phi_1 \oplus \Psi_1 = n} (103x_1^2 - 13(\Phi_1 \oplus \Psi_1))q^n \quad (0.12) \\ &= \frac{1}{103} (154q^2 - 104q^4 + 100q^6 - 182q^7 \\ & \quad + 408q^8 - 56q^9 - 108q^{10} + 1024q^{12} \\ & \quad - 338q^{13} - 522q^{14} + 868q^{15} - 230q^{16} \\ & \quad - 1326q^{17} + 338q^{18} - 246q^{19} - 1668q^{20} \\ & \quad - 1772q^{21} + 2336q^{22} - 106q^{23} - 436q^{24} \\ & \quad + 2756q^{25} + \dots), \end{aligned}$$

$$\begin{aligned} & \Theta_{\Phi_1 \oplus \Psi_1, \varphi_{12}}(q) \\ &= \frac{1}{2 \cdot 103} \sum_{n=1}^{\infty} \sum_{\Phi_1 \oplus \Psi_1 = n} (2 \cdot 103x_1x_2 + (\Phi_1 \oplus \Psi_1))q^n \quad (0.13) \\ &= \frac{1}{103} (2q^2 + 4q^4 + 12q^6 + 7q^7 + 16q^8 \\ & \quad + 18q^9 + 20q^{10} + 24q^{12} + 13q^{13} \\ & \quad - 178q^{14} + 30q^{15} + 302q^{16} + 51q^{17} \\ & \quad - 322q^{18} - 355q^{19} + 492q^{20} \\ & \quad - 328q^{21} - 280q^{22} + 115q^{23} + 508q^{24} \\ & \quad + 100q^{25} + \dots), \end{aligned}$$

$$\begin{aligned} & \Theta_{\Phi_1 \oplus \Psi_1, \varphi_{22}}(q) \\ &= \frac{1}{103^2} \sum_{n=1}^{\infty} \sum_{\Phi_1 \oplus \Psi_1 = n} (103^2x_2^2 - 182(\Phi_1 \oplus \Psi_1))q^n \quad (0.14) \\ &= \frac{1}{103^2} (-728q^2 - 1456q^4 - 4368q^6 \\ & \quad - 2548q^7 - 5824q^8 \\ & \quad - 6552q^9 - 7280q^{10} - 8736q^{12} \\ & \quad + 16486q^{13} + 11026q^{14} - 10920q^{15} \\ & \quad - 13726q^{16} + 23872q^{17} \\ & \quad + 9676q^{18} + 470q^{19} \\ & \quad + 55752q^{20} + 54296q^{21} - 5612q^{22} \\ & \quad + 64230q^{23} + 7492q^{24} \\ & \quad - 36400q^{25} + \dots), \end{aligned}$$

$$\begin{aligned} & \Theta_{\Phi_1 \oplus \Psi_1, \varphi_{23}}(q) \\ &= \frac{1}{103^2} \sum_{n=1}^{\infty} \sum_{\Phi_1 \oplus \Psi_1 = n} (103^2x_2x_3 + 28(\Phi_1 \oplus \Psi_1))q^n \quad (0.15) \\ &= \frac{1}{103^2} (112q^2 + 224q^4 + 672q^6 + 392q^7 \\ & \quad + 896q^8 + 1008q^9 \\ & \quad + 1120q^{10} + 1344q^{12} + 728q^{13} \\ & \quad + 1568q^{14} + 1680q^{15} + 5376q^{16} \\ & \quad + 2856q^{17} + 5040q^{18} \\ & \quad + 3192q^{19} + 4480q^{20} \\ & \quad + 4704q^{21} + 7392q^{22} + 6440q^{23} \\ & \quad + 5376q^{24} + 5600q^{25} + \dots), \end{aligned}$$

$$\begin{aligned} & \Theta_{\Phi_1 \oplus \Psi_1, \varphi_{33}}(q) \\ &= \frac{1}{103} \sum_{n=1}^{\infty} \sum_{\Phi_1 \oplus \Psi_1 = n} (103x_3^2 - 7(\Phi_1 \oplus \Psi_1))q^n \quad (0.16) \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{103} (-28q^2 + 150q^4 + 244q^6 - 98q^7 \\
 &\quad - 18q^8 - 252q^9 + 132q^{10} \\
 &\quad + 76q^{12} \\
 &\quad - 182q^{13} - 186q^{14} - 420q^{15} + 304q^{16} \\
 &\quad + 522q^{17} + 800q^{18} \\
 &\quad + 850q^{19} \\
 &\quad - 708q^{20} - 764q^{21} - 612q^{22} - 1198q^{23} \\
 &\quad + 716q^{24} + 248q^{25} + \dots),
 \end{aligned}$$

$$\begin{aligned}
 &\Theta_{\Phi_1 \oplus \Psi_1, \varphi_{34}}(q) \\
 &= \frac{1}{2 \cdot 103} \sum_{n=1}^{\infty} \sum_{\Phi_1 \oplus \Psi_1 = n} (2 \quad (0.17) \\
 &\quad \cdot 103x_3x_4 + (\Phi_1 \oplus \Psi_1)) q^n \\
 &= \frac{1}{103} (2q^2 + 4q^4 + 12q^6 + 7q^7 - 190q^8 \\
 &\quad + 18q^9 - 392q^{10} + 24q^{12} \\
 &\quad + 13q^{13} \\
 &\quad + 234q^{14} + 30q^{15} + 96q^{16} - 361q^{17} \\
 &\quad + 90q^{18} - 767q^{19} + 80q^{20} \\
 &\quad - 328q^{21} + 132q^{22} + 115q^{23} - 316q^{24} \\
 &\quad - 724q^{25} + \dots),
 \end{aligned}$$

and

$$\begin{aligned}
 &\Theta_{\Phi_1 \oplus \Psi_1, \varphi_{44}}(q) \\
 &= \frac{1}{103} \sum_{n=1}^{\infty} \sum_{\Phi_1 \oplus \Psi_1 = n} (103x_4^2 \quad (0.18) \\
 &\quad - 4(\Phi_1 \oplus \Psi_1)) q^n \\
 &= \frac{1}{103} (-8q^2 - 8q^4 - 16q^6 + 198q^7 \\
 &\quad + 190q^8 + 396q^9 + 396q^{10} \\
 &\quad - 16q^{12} \\
 &\quad - 8q^{13} + 190q^{14} + 396q^{15} + 776q^{16} \\
 &\quad + 182q^{17} - 40q^{18} + 388q^{19} \\
 &\quad + 380q^{20}
 \end{aligned}$$

$$\begin{aligned}
 &+ 792q^{21} + 776q^{22} + 372q^{23} + 380q^{24} \\
 &\quad + 792q^{25} + \dots).
 \end{aligned}$$

The rank of the matrix of these coefficients of the theta series is 20 and the set of theta series in the Theorem is a basis of the subspace of $S_4(\Gamma_0(103))$ spanned by the generalized theta series of the form 2.4 induced by spherical functions of the form 2.3.

1. The solutions of $Q=n$ and the Generalized Theta Series associated to the Quadratic forms

By solving the equations

$$F_1 = x_1^2 + x_1x_2 + 26x_2^2 = n,$$

$$\Phi_1 = 2x_1^2 + x_1x_2 + 13x_2^2 = n,$$

and

$$\Psi_1 = 4x_1^2 + 3x_1x_2 + 7x_2^2 = n$$

for

$$n = 1, 2, \dots, 25,$$

we get

$$\Theta_{F_1}(q) = 1 + 2q + 2q^4 + 2q^9 + 2q^{16} + 2q^{25} + \dots,$$

$$\Theta_{\Phi_1}(q) = 1 + 2q^2 + 2q^8 + 2q^{13} + 2q^{14} + 2q^{16} + 2q^{18} + 2q^{19} + 2q^{23} + \dots$$

$$\Theta_{\Psi_1}(q) = 1 + 2q^4 + 2q^7 + 2q^8 + 2q^{14} + 2q^{16} + 2q^{17} + \dots,$$

$$\Theta_{F_2}(q) = \Theta_{F_1}(q)\Theta_{\Phi_1}(q) = 1 + 4q + 4q^2 + 4q^4 + 8q^5 + 4q^8 + 4q^9 + 8q^{10} + 8q^{13}$$

$$+ 4q^{16} + 8q^{17} + 4q^{18} + 8q^{20} + 12q^{25} + \dots,$$

$$\Theta_{F_3}(q) = \Theta_{F_2}(q)\Theta_{\Psi_1}(q) = 1 + 6q + 12q^2 + 8q^3 + 6q^4 + 24q^5 + 24q^6 + 12q^8 + 30q^9 + 24q^{10} + 24q^{11}$$

$$\begin{aligned}
&+8q^{12} + 24q^{13} + 48q^{14} + 6q^{16} + 48q^{17} \\
&\quad + 36q^{18} + 24q^{19} + 24q^{20} \\
&\quad + 48q^{21} + 24q^{22} + 24q^{24} \\
&\quad + 30q^{25} + \dots,
\end{aligned}$$

$$\begin{aligned}
\Theta_{F_4}(q) &= \Theta_{F_2}(q)\Theta_{F_2}(q) \\
&= 1 + 8q + 24q^2 + 32q^3 \\
&\quad + 24q^4 + 48q^5 + 96q^6 \\
&\quad + 64q^7 + 24q^8
\end{aligned}$$

$$\begin{aligned}
104q^9 + 144q^{10} + 96q^{11} + 96q^{12} \\
+ 112q^{13} + 192q^{14} \\
+ 192q^{15} + 24q^{16} + 144q^{17}
\end{aligned}$$

$$\begin{aligned}
+312q^{18} + 160q^{19} + 144q^{20} + 256q^{21} \\
+ 288q^{22} + 192q^{23} + 96q^{24} \\
+ 248q^{25} + \dots,
\end{aligned}$$

$$\begin{aligned}
\Theta_{\Phi_2}(q) &= \Theta_{\Phi_1}(q)\Theta_{\Phi_1}(q) \\
&= 1 + 4q^2 + 4q^4 + 4q^8 \\
&\quad + 8q^{10} + 4q^{13} + 4q^{14}
\end{aligned}$$

$$\begin{aligned}
+8q^{15} + 16q^{16} + 12q^{18} + 4q^{19} + 8q^{20} \\
+ 16q^{21} + 8q^{22} + 4q^{23} \\
+ 8q^{24} + 8q^{25} + \dots,
\end{aligned}$$

$$\begin{aligned}
\Theta_{F_4}(q) &= \Theta_{F_2}(q)\Theta_{F_2}(q) \\
&= 1 + 8q + 24q^2 + 32q^3 \\
&\quad + 24q^4 + 48q^5 + 96q^6 \\
&\quad + 64q^7 + 24q^8
\end{aligned}$$

$$\begin{aligned}
104q^9 + 144q^{10} + 96q^{11} + 96q^{12} \\
+ 112q^{13} + 192q^{14} \\
+ 192q^{15} + 24q^{16} + 144q^{17}
\end{aligned}$$

$$\begin{aligned}
+312q^{18} + 160q^{19} + 144q^{20} + 256q^{21} \\
+ 288q^{22} + 192q^{23} + 96q^{24} \\
+ 248q^{25} + \dots,
\end{aligned}$$

$$\begin{aligned}
\Theta_{\Phi_2}(q) &= \Theta_{\Phi_1}(q)\Theta_{\Phi_1}(q) \\
&= 1 + 4q^2 + 4q^4 + 4q^8 \\
&\quad + 8q^{10} + 4q^{13} + 4q^{14}
\end{aligned}$$

$$\begin{aligned}
+8q^{15} + 16q^{16} + 12q^{18} + 4q^{19} + 8q^{20} \\
+ 16q^{21} + 8q^{22} + 4q^{23} \\
+ 8q^{24} + 8q^{25} + \dots,
\end{aligned}$$

$$\begin{aligned}
\Theta_{\Phi_3}(q) &= \Theta_{\Phi_2}(q)\Theta_{\Phi_1}(q) \\
&= 1 + 6q^2 + 12q^4 + 8q^6 \\
&\quad + 6q^8 + 24q^{10} + 24q^{12} \\
&\quad + 6q^{13}
\end{aligned}$$

$$\begin{aligned}
+6q^{14} + 24q^{15} + 42q^{16} + 24q^{17} + 78q^{18} \\
+ 6q^{19} + 48q^{20} + 48q^{21} \\
+ 48q^{22} + 78q^{23} + 80q^{24} \\
+ 24q^{25} + \dots,
\end{aligned}$$

$$\begin{aligned}
\Theta_{\Phi_4}(q) &= \Theta_{\Phi_2}(q)\Theta_{\Phi_2}(q) \\
&= 1 + 8q^2 + 24q^4 + 32q^6 \\
&\quad + 24q^8 + 48q^{10} + 96q^{12} \\
&\quad + 8q^{13} + 72q^{14}
\end{aligned}$$

$$\begin{aligned}
+48q^{15} + 80q^{16} + 96q^{17} + 248q^{18} \\
+ 72q^{19} + 304q^{20} + 96q^{21} \\
+ 208q^{22} + 296q^{23} \\
+ 336q^{24} + 304q^{25} + \dots,
\end{aligned}$$

$$\begin{aligned}
\Theta_{\Psi_2}(q) &= \Theta_{\Psi_1}(q)\Theta_{\Psi_1}(q) \\
&= 1 + 4q^4 + 4q^7 + 8q^8 \\
&\quad + 8q^{11} + 8q^{12}
\end{aligned}$$

$$\begin{aligned}
+8q^{14} + 8q^{15} + 8q^{16} + 4q^{17} + 8q^{18} \\
+ 8q^{20} + 16q^{21} + 8q^{22} \\
+ 8q^{23} + 16q^{24} + 8q^{25} + \dots,
\end{aligned}$$

$$\begin{aligned}
\Theta_{\Psi_3}(q) &= \Theta_{\Psi_2}(q)\Theta_{\Psi_1}(q) \\
&= 1 + 6q^4 + 6q^7 + 18q^8 \\
&\quad + 24q^{11} + 32q^{12} + 18q^{14} \\
&\quad + 48q^{15}
\end{aligned}$$

$$\begin{aligned}
+42q^{16} + 6q^{17} + 48q^{18} + 48q^{19} + 48q^{20} \\
+ 56q^{21} + 72q^{22} + 48q^{23} \\
+ 80q^{24} + 96q^{25} + \dots,
\end{aligned}$$

$$\begin{aligned}\Theta_{\Psi_4}(q) &= \Theta_{\Psi_2}(q)\Theta_{\Psi_2}(q) \\ &= 1 + 8q^4 + 8q^7 + 32q^8 \\ &\quad + 48q^{11} + 80q^{12} + 32q^{14} \\ &\quad + 144q^{15}\end{aligned}$$

$$\begin{aligned}&+ 144q^{16} + 8q^{17} + 144q^{18} + 256q^{19} \\ &\quad + 208q^{20} + 128q^{21} \\ &\quad + 336q^{22} + 336q^{23} \\ &\quad + 320q^{24} + 400q^{25} + \dots,\end{aligned}$$

$$\begin{aligned}\Theta_{F_1 \oplus \Phi_1}(q) &= \Theta_{F_1}(q)\Theta_{\Phi_1}(q) \\ &= 1 + 2q + 2q^2 + 4q^3 + 2q^4 \\ &\quad + 4q^6 + 2q^8 + 6q^9 + 4q^{11} \\ &\quad + 4q^{12}\end{aligned}$$

$$\begin{aligned}&+ 2q^{13} + 6q^{14} + 4q^{15} + 6q^{16} + 16q^{17} \\ &\quad + 10q^{18} + 6q^{19} + 12q^{20} \\ &\quad + 8q^{22} + 10q^{23} + 8q^{24} \\ &\quad + 10q^{25} + \dots,\end{aligned}$$

$$\begin{aligned}\Theta_{F_3 \oplus \Phi_1}(q) &= \Theta_{F_3}(q)\Theta_{\Phi_1}(q) \\ &= 1 + 6q + 14q^2 + 20q^3 \\ &\quad + 30q^4 + 40q^5 + 36q^6\end{aligned}$$

$$\begin{aligned}&48q^7 + 62q^8 + 42q^9 + 72q^{10} + 100q^{11} \\ &\quad + 68q^{12} + 122q^{13} + 126q^{14} \\ &\quad + 84q^{16}\end{aligned}$$

$$\begin{aligned}&+ 148q^{17} + 182q^{18} + 294q^{19} + 208q^{20} \\ &\quad + 256q^{21} + 328q^{22} \\ &\quad + 266q^{23} + 312q^{24} \\ &\quad + 322q^{25} + \dots,\end{aligned}$$

$$\begin{aligned}\Theta_{F_2 \oplus \Phi_2}(q) &= \Theta_{F_2}(q)\Theta_{\Phi_2}(q) \\ &= 1 + 4q + 8q^2 + 16q^3 \\ &\quad + 24q^4 + 24q^5 + 32q^6\end{aligned}$$

$$\begin{aligned}&+ 32q^7 + 24q^8 + 52q^9 + 48q^{10} + 48q^{11} \\ &\quad + 96q^{12} + 60q^{13} + 84q^{14} \\ &\quad + 136q^{15} + 84q^{16}\end{aligned}$$

$$\begin{aligned}&+ 168q^{17} + 208q^{18} + 180q^{19} + 304q^{20} \\ &\quad + 272q^{21} + 232q^{22} \\ &\quad + 356q^{23} + 296q^{24} \\ &\quad + 356q^{25} + \dots,\end{aligned}$$

$$\begin{aligned}\Theta_{F_1 \oplus \Phi_3}(q) &= \Theta_{F_1}(q)\Theta_{\Phi_3}(q) \\ &= 1 + 2q + 6q^2 + 12q^3 \\ &\quad + 14q^{14} + 24q^5 + 20q^6 \\ &\quad + 16q^7 + 30q^8\end{aligned}$$

$$\begin{aligned}&+ 14q^9 + 40q^{10} + 60q^{11} + 36q^{12} + 78q^{13} \\ &\quad + 66q^{14} + 52q^{15} + 146q^{16}\end{aligned}$$

$$\begin{aligned}&+ 132q^{17} + 150q^{18} + 258q^{19} + 168q^{20} \\ &\quad + 240q^{21} + 328q^{22} \\ &\quad + 198q^{23} + 392q^{24} \\ &\quad + 366q^{25} + \dots,\end{aligned}$$

$$\begin{aligned}\Theta_{F_1 \oplus \Psi_1}(q) &= \Theta_{F_1}(q)\Theta_{\Psi_1}(q) \\ &= 1 + 2q + 4q^4 + 4q^5 + 2q^7 \\ &\quad + 10q^8 + 6q^9 + 4q^{11} + 4q^{12}\end{aligned}$$

$$\begin{aligned}&+ 6q^{13} + 6q^{14} + 4q^{15} + 8q^{16} + 14q^{17} \\ &\quad + 8q^{18} + 6q^{18} + 8q^{20} \\ &\quad + 4q^{21} + 4q^{22} + 8q^{23} \\ &\quad + 4q^{24} + 6q^{25} + \dots,\end{aligned}$$

$$\begin{aligned}\Theta_{F_3 \oplus \Psi_1}(q) &= \Theta_{F_3}(q)\Theta_{\Psi_1}(q) \\ &= 1 + 6q + 12q^2 + 8q^3 \\ &\quad + 8q^4 + 36q^5 + 48q^6 \\ &\quad + 18q^7\end{aligned}$$

$$\begin{aligned}&+ 38q^8 + 114q^9 + 112q^{10} + 52q^{11} \\ &\quad + 92q^{12} + 180q^{13} + 146q^{14} \\ &\quad + 84q^{15} + 132q^{16} \\ &\quad + 234q^{17} + 276q^{18} + 176q^{19} + 176q^{20} \\ &\quad + 348q^{21} + 312q^{22} \\ &\quad + 168q^{23} + 252q^{24} \\ &\quad + 426q^{25} + \dots,\end{aligned}$$

$$\begin{aligned}
\Theta_{F_2 \oplus \Psi_2}(q) &= \Theta_{F_2}(q)\Theta_{\Psi_2}(q) \\
&= 1 + 4q + 4q^2 + 8q^4 \\
&\quad + 24q^5 + 16q^6 + 4q^7 \\
&\quad + 44q^8 \\
&\quad + 84q^9 + 40q^{10} + 24q^{11} + 120q^{12} \\
&\quad\quad + 152q^{13} + 72q^{14} + 88q^{15} \\
&\quad\quad + 220q^{16} + 236q^{17} \\
&\quad + 156q^{18} + 176q^{19} + 256q^{20} + 320q^{21} \\
&\quad\quad + 280q^{22} + 248q^{23} \\
&\quad\quad + 400q^{24} + 500q^{25} + \dots, \\
\Theta_{F_1 \oplus \Psi_3}(q) &= \Theta_{F_1}(q)\Theta_{\Psi_3}(q) \\
&= 1 + 2q + 8q^4 + 12q^5 \\
&\quad + 6q^7 + 42q^8 + 38q^9 \\
&\quad + 36q^{11} \\
&\quad + 116q^{12} + 76q^{13} + 18q^{14} + 132q^{15} \\
&\quad\quad + 216q^{16} + 126q^{17} \\
&\quad + 96q^{18} + 240q^{19} + 288q^{20} + 228q^{21} \\
&\quad\quad + 280q^{22} + 336q^{23} \\
&\quad\quad + 404q^{24} + 454q^{25} + \dots, \\
\Theta_{\Phi_1 \oplus \Psi_1}(q) &= \Theta_{\Phi_1}(q)\Theta_{\Psi_1}(q) \\
&= 1 + 2q^2 + 2q^4 + 4q^6 \\
&\quad + 2q^7 + 4q^8 + 4q^9 + 4q^{10} \\
&\quad + 4q^{12} + 4q^{13} + 4q^{14} \\
&\quad + 8q^{15} + 14q^{16} + 6q^{17} + 10q^{18} + 6q^{19} \\
&\quad\quad + 12q^{20} + 12q^{21} + 12q^{22} \\
&\quad\quad + 14q^{23} + 12q^{24} + 8q^{25} \\
&\quad\quad + \dots, \\
\Theta_{\Phi_3 \oplus \Psi_1}(q) &= \Theta_{\Phi_3}(q)\Theta_{\Psi_1}(q) \\
&= 1 + 6q^2 + 14q^4 + 20q^6 \\
&\quad + 2q^7 + 32q^8 + 12q^9 \\
&\quad + 52q^{10} \\
&\quad + 24q^{11} + 60q^{12} + 22q^{13} + 72q^{14} + 36q^{15} \\
&\quad\quad + 116q^{16} + 86q^{17} + 174q^{18} \\
&\quad + 114q^{19} + 232q^{20} + 144q^{21} + 292q^{22} \\
&\quad\quad + 238q^{23} + 368q^{24} \\
&\quad\quad + 336q^{25} + \dots, \\
\Theta_{\Phi_2 \oplus \Psi_2}(q) &= \Theta_{\Phi_2}(q)\Theta_{\Psi_2}(q) \\
&= 1 + 4q^2 + 8q^4 + 16q^6 \\
&\quad + 4q^7 + 28q^8 + 16q^9 \\
&\quad + 40q^{10} \\
&\quad + 24q^{11} + 56q^{12} + 36q^{13} + 76q^{14} + 64q^{15} \\
&\quad\quad + 120q^{16} + 84q^{17} + 164q^{18} \\
&\quad + 116q^{19} + 192q^{20} + 160q^{21} + 288q^{22} \\
&\quad\quad + 252q^{23} + 376q^{24} \\
&\quad\quad + 368q^{25} + \dots, \\
\Theta_{\Phi_1 \oplus \Psi_3}(q) &= \Theta_{\Phi_1}(q)\Theta_{\Psi_3}(q) \\
&= 1 + 2q^2 + 6q^4 + 12q^6 \\
&\quad + 6q^7 + 20q^8 + 12q^9 \\
&\quad + 36q^{10} \\
&\quad + 24q^{11} + 44q^{12} + 50q^{13} + 84q^{14} + 60q^{15} \\
&\quad\quad + 116q^{16} + 114q^{17} \\
&\quad\quad + 146q^{18} \\
&\quad + 110q^{19} + 232q^{20} + 200q^{21} + 252q^{22} \\
&\quad\quad + 282q^{23} + 392q^{24} \\
&\quad\quad + 328q^{25} + \dots, \\
\Theta_{F_2 \oplus \Phi_1 \oplus \Psi_1}(q) &= \Theta_{F_2}(q) \cdot \Theta_{\Phi_1 \oplus \Psi_1}(q) \\
&= 1 + 4q + 6q^2 + 8q^3 \\
&\quad + 14q^4 + 16q^5 + 20q^6 \\
&\quad + 34q^7 \\
&\quad + 40q^8 + 48q^9 + 68q^{10} + 80q^{11} + 76q^{12} \\
&\quad\quad + 82q^{13} + 108q^{14} + 100q^{15} \\
&\quad\quad + 120q^{16} \\
&\quad + 182q^{17} + 190q^{18} + 214q^{19} + 240q^{20} \\
&\quad\quad + 256q^{21} + 292q^{22} \\
&\quad\quad + 314q^{23} + 336q^{24} \\
&\quad\quad + 380q^{25} + \dots,
\end{aligned}$$

$$\begin{aligned} \Theta_{F_1 \oplus \Phi_2 \oplus \Psi_1}(q) &= \Theta_{\Phi_2}(q) \cdot \Theta_{F_1 \oplus \Psi_1}(q) \\ &= 1 + 2q + 4q^2 + 8q^3 + 8q^4 \\ &\quad + 12q^5 + 16q^6 + 18q^7 \\ &\quad + 30q^8 + 38q^9 + 48q^{10} + 52q^{11} + 60q^{12} \\ &\quad + 64q^{13} + 62q^{14} + 92q^{15} \\ &\quad + 104q^{16} \\ &\quad + 130q^{17} + 172q^{18} + 196q^{19} + 208q^{20} \\ &\quad + 236q^{21} + 272q^{22} \\ &\quad + 292q^{23} + 372q^{24} \\ &\quad + 398q^{25} + \dots, \end{aligned}$$

$$\begin{aligned} \Theta_{F_1 \oplus \Phi_1 \oplus \Psi_2}(q) &= \Theta_{\Psi_2}(q) \cdot \Theta_{F_1 \oplus \Phi_1}(q) \\ &= 1 + 2q + 2q^2 + 4q^3 + 6q^4 \\ &\quad + 8q^5 + 12q^6 \\ &\quad + 20q^7 + 26q^8 + 30q^9 + 48q^{10} + 52q^{11} \\ &\quad + 52q^{12} + 74q^{13} + 94q^{14} \\ &\quad + 100q^{15} + 116q^{16} \\ &\quad + 168q^{17} \\ &\quad + 162q^{18} + 158q^{19} + 216q^{20} + 240q^{21} \\ &\quad + 264q^{22} + 314q^{23} \\ &\quad + 392q^{24} + 414q^{25} + \dots. \end{aligned}$$

Remark 1.

It is known that

$$\begin{aligned} L_{\Theta_{F_1}}(s) &= 2 \sum_{a \text{ integral ideal} \in A_{F_1}} N(a)^{-s}, \text{ where} \end{aligned}$$

$$A_{F_1} = \left[\left[1, \frac{1+i\sqrt{103}}{2} \right] \right] \text{ is the ideal class corresponding to } F_1,$$

$$\begin{aligned} L_{\Theta_{\Phi_1}}(s) &= 2 \sum_{a \text{ integral ideal} \in A_{\Phi_1}} N(a)^{-s}, \text{ where} \end{aligned}$$

$$\begin{aligned} A_{\Phi_1} &= \left[\left[2, 1 \right. \right. \\ &\quad \left. \left. + \frac{1+i\sqrt{103}}{2} \right] \right] \text{ is the ideal class corresponding to } \Phi_1, \end{aligned}$$

$$\begin{aligned} L_{\Theta_{\Psi_1}}(s) &= 2 \sum_{a \text{ integral ideal} \in A_{\Psi_1}} N(a)^{-s}, \text{ where} \end{aligned}$$

$$\begin{aligned} A_{\Psi_1} &= \left[\left[4, -2 \right. \right. \\ &\quad \left. \left. + \frac{1+i\sqrt{103}}{2} \right] \right] \text{ is the ideal class corresponding to } \Psi_1. \end{aligned}$$

In general, if ρ_k is a character of the class group of $\mathbb{Q}(\sqrt{-103})$, i.e.,

$$\begin{aligned} \rho_k(\Phi_1^m) &= e^{2\pi i m k / 5} \text{ for } m = 0, 1, 2, 3, 4, k \\ &= 0, 1, 2, 3, 4, \end{aligned}$$

Remark 2. The Hecke L-functions for these characters will be

$$\begin{aligned} L_{\rho_k}(s) &= L_{\Theta_{F_1}}(s) + \rho_k(\Phi_1)L_{\Theta_{\Phi_1}}(s) \\ &\quad + \rho_k(\Phi_1^2)L_{\Theta_{\Psi_1'}}(s) \\ &\quad + \rho_k(\Phi_1^3)L_{\Theta_{\Psi_1}}(s) \\ &\quad + \rho_k(\Phi_1^4)L_{\Theta_{\Phi_1'}}(s) \end{aligned}$$

$$\begin{aligned} &= L_{\Theta_{F_1}}(s) + (\rho_k(\Phi_1) + \rho_k(\Phi_1^4))L_{\Theta_{\Phi_1}}(s) \\ &\quad + (\rho_k(\Phi_1^2) \\ &\quad + \rho_k(\Phi_1^3))L_{\Theta_{\Psi_1}}(s) \end{aligned}$$

$$\begin{aligned} &L_{\Theta_{F_1}}(s) + (\rho_k(\Phi_1) + \rho_k(\Phi_1)^{-1})L_{\Theta_{\Phi_1}}(s) \\ &\quad + (\rho_k(\Phi_1^2) \\ &\quad + \rho_k(\Phi_1^2)^{-1})L_{\Theta_{\Psi_1}}(s) \end{aligned}$$

which are the L-functions of weight 1 modular forms

$$f_{\rho_k}(q) = \frac{1}{2} \left(\Theta_{F_1}(q) + (\rho_k(\Phi_1) + \rho_k(\Phi'_1))\Theta_{\Phi_1}(q) + (\rho_k(\Psi_1) + \rho_k(\Psi'_1))\Theta_{\Psi_1}(q) \right)$$

for $k = 0, 1, 2, 3, 4$.

Because of the unique factorization of the ideals in $\mathbb{Q}(\sqrt{-103})$, these L functions has Euler product, hence the modular forms f_{ρ_k} are Hecke eigenforms. In particular, the L function of the Hecke eigenform of weight 1

$$f_{\rho_0}(q) := \frac{1}{2} \left(\Theta_{F_1}(q) + 2\Theta_{\Phi_1}(q) + 2\Theta_{\Psi_1}(q) \right)$$

$$= \frac{1}{2} \left((1 + 2q + 2q^4 + 2q^9 + 2q^{16} + 2q^{25} + \dots) + \right.$$

$$\left. 2(1 + 2q^2 + 2q^8 + 2q^{13} + 2q^{14} + 2q^{16} + 2q^{16} + 2q^{18} + 2q^{19} + 2q^{23} + \dots) \right)$$

$$+ 2(1 + 2q^4 + 2q^7 + 2q^8 + 2q^{14} + 2q^{13} + 2q^{16} + 2q^{17} + \dots)$$

$$= \frac{5}{2} + q + 4q^2 + 5q^4 + 4q^7 + 8q^8 + 8q^{13} + 8q^{14} + 9q^{16} + 4q^{17} + 4q^{18} + \dots$$

is exactly the Dedekind zeta function of $\mathbb{Q}(\sqrt{-103})$ which is equal to

$$\zeta(s)L(\chi, s),$$

where $\chi(d) = \left(\frac{-103}{d}\right)$ is the natural Dirichlet character associated with the quadratic field $\mathbb{Q}(\sqrt{-103})$ [7]. In fact, f_{ρ_0} is the Eisenstein series of weight 1. From here, we simply obtain the formula for the sum of the number of representations of an integer n as

$$r(F_1, n) + r(\Phi_1, n) + r(\Phi'_1, n) + r(\Psi_1, n) + r(\Psi'_1, n)$$

$$= r(F_1, n) + 2r(\Phi_1, n) + 2r(\Psi_1, n) = 2 \sum_{d|n} \left(\frac{-103}{d}\right),$$

where $\left(\frac{D}{d}\right)$ is the Kronecker symbol.

By [7], it is also known that for $m = 0, 1, 2, 3, 4$,

$$\sum_{k=0}^4 \rho_k(\Phi_1^0) L_{\rho_k}(s) = \frac{2 \cdot 5}{2} \left(\frac{\sqrt{103}}{2}\right)^{-s} \zeta(2s) \frac{\pi^s}{\Gamma(s)} E\left(-\frac{1}{2} + \frac{i\sqrt{103}}{2}, s\right)$$

$$\sum_{k=0}^4 \rho_k(\Phi_1) L_{\rho_k}(s) = \frac{2 \cdot 5}{2} \left(\frac{\sqrt{103}}{2}\right)^{-s} \zeta(2s) \frac{\pi^s}{\Gamma(s)} E\left(-\frac{1}{4} + \frac{i\sqrt{103}}{4}, s\right)$$

$$\sum_{k=0}^4 \rho_k(\Phi_1^2) L_{\rho_k}(s) = \frac{2 \cdot 5}{2} \left(\frac{\sqrt{103}}{2}\right)^{-s} \zeta(2s) \frac{\pi^s}{\Gamma(s)} E\left(\frac{3}{8} + \frac{i\sqrt{103}}{8}, s\right)$$

$$\sum_{k=0}^4 \rho_k(\Phi_1^3) L_{\rho_k}(s) = \frac{2 \cdot 5}{2} \left(\frac{\sqrt{103}}{2}\right)^{-s} \zeta(2s) \frac{\pi^s}{\Gamma(s)} E\left(-\frac{3}{8} + \frac{i\sqrt{103}}{8}, s\right)$$

$$\sum_{k=0}^4 \rho_k(\Phi_1^k) L_{\rho_k}(s) = \frac{2 \cdot 5}{2} \left(\frac{\sqrt{103}}{2} \right)^{-s} \zeta(2s) \frac{\pi^s}{\Gamma(s)} E\left(\frac{1}{4} + \frac{i\sqrt{103}}{4}, s \right),$$

where

$$E(z, s) = \frac{1}{2} \pi^{-s} \Gamma(s) \sum_{(m,n) \in \mathbb{Z} \times \mathbb{Z} \setminus \{(0,0)\}} \frac{y^s}{|mz + n|^{2s}} \quad z \in H, s \in \mathbb{C}$$

is the nonholomorphic Eisenstein series. $L_{\rho_k}(s)$ has analytic continuation to the whole complex plane as an entire function except for a simple pole at 1 for ρ_0 with

$$res_{s=1} L_{\rho_0} = res_{s=1} \zeta_K = \frac{2\pi \cdot 5}{2\sqrt{103}} = \frac{5\pi}{\sqrt{103}}$$

and its completion, i.e.,

$$\Lambda_{\rho_k}(s) = (2\pi)^{-s} \Gamma(s) 103^{s/2} L_{\rho_k}(s)$$

is equal to

$$\begin{aligned} &= \zeta(2s) (\rho_k(\Phi_1^0) E\left(-\frac{1}{2} + \frac{i\sqrt{103}}{2}, s\right) \\ &\quad + \rho_k(\Phi_1) E\left(-\frac{1}{4} + \frac{i\sqrt{103}}{4}, s\right) \\ &\quad + \rho_k(\Phi_1^2) E\left(\frac{3}{8} + \frac{i\sqrt{103}}{8}, s\right) \\ &\quad + \rho_k(\Phi_1^3) E\left(-\frac{3}{8} + \frac{i\sqrt{103}}{8}, s\right) \\ &\quad + \rho_k(\Phi_1^4) E\left(\frac{1}{4} + \frac{i\sqrt{103}}{4}, s\right) \end{aligned}$$

and satisfies the functional equation

$$\Lambda_{\rho_k}(s) = \Lambda_{\rho_k}(1-s) \text{ for all } s \neq 1.$$

1. Representation Numbers of n

Theorem 1. The difference between the following theta series of the quadratic forms

$$\begin{aligned} \Theta_{F_4}(q) &= \Theta_{F_2}(q) \Theta_{F_2}(q) \\ &= 1 + 8q + 24q^2 + 32q^3 \\ &\quad + 24q^4 + 48q^5 + 96q^6 \\ &\quad + 64q^7 + 24q^8 \end{aligned}$$

$$\begin{aligned} &104q^9 + 144q^{10} + 96q^{11} + 96q^{12} \\ &\quad + 112q^{13} + 192q^{14} \\ &\quad + 192q^{15} + 24q^{16} + 144q^{17} \end{aligned}$$

$$\begin{aligned} &+ 312q^{18} + 160q^{19} + 144q^{20} + 256q^{21} \\ &\quad + 288q^{22} + 192q^{23} + 96q^{24} \\ &\quad + 248q^{25} + \dots, \end{aligned}$$

$$\begin{aligned} \Theta_{\Phi_4}(q) &= \Theta_{\Phi_2}(q) \Theta_{\Phi_2}(q) \\ &= 1 + 8q^2 + 24q^{24} + 32q^6 \\ &\quad + 24q^8 + 48q^{10} + 96q^{12} \\ &\quad + 8q^{13} + 72q^{14} \end{aligned}$$

$$\begin{aligned} &+ 48q^{15} + 80q^{16} + 96q^{17} + 248q^{18} \\ &\quad + 72q^{19} + 304q^{20} + 96q^{21} \\ &\quad + 208q^{22} + 296q^{23} \\ &\quad + 336q^{24} + 304q^{25} + \dots, \end{aligned}$$

$$\begin{aligned} \Theta_{\Psi_4}(q) &= \Theta_{\Psi_2}(q) \Theta_{\Psi_2}(q) \\ &= 1 + 8q^4 + 8q^7 + 32q^8 \\ &\quad + 48q^{11} + 80q^{12} + 32q^{14} \\ &\quad + 144q^{15} \end{aligned}$$

$$\begin{aligned} &+ 144q^{16} + 8q^{17} + 144q^{18} + 256q^{19} \\ &\quad + 208q^{20} + 128q^{21} \\ &\quad + 336q^{22} + 336q^{23} \\ &\quad + 320q^{24} + 400q^{25} + \dots, \end{aligned}$$

$$\begin{aligned} \Theta_{F_3 \oplus \Phi_1}(q) &= \Theta_{F_3}(q) \Theta_{\Phi_1}(q) \\ &= 1 + 6q + 14q^2 + 20q^3 \\ &\quad + 30q^4 + 40q^5 + 36q^6 \end{aligned}$$

$$\begin{aligned} &48q^7 + 62q^8 + 42q^9 + 72q^{10} + 100q^{11} \\ &\quad + 68q^{12} + 122q^{13} + 126q^{14} \\ &\quad + 168q^{16} \end{aligned}$$

$$\begin{aligned}
&+148q^{17} + 182q^{18} + 294q^{19} + 208q^{20} \\
&\quad + 256q^{21} + 328q^{22} \\
&\quad + 266q^{23} + 312q^{24} \\
&\quad + 322q^{25} + \dots,
\end{aligned}$$

$$\begin{aligned}
\Theta_{F_2 \oplus \Phi_2}(q) &= \Theta_{F_2}(q)\Theta_{\Phi_2}(q) \\
&= 1 + 4q + 8q^2 + 16q^3 \\
&\quad + 24q^4 + 24q^5 + 32q^6
\end{aligned}$$

$$\begin{aligned}
&+32q^7 + 24q^8 + 52q^9 + 48q^{10} + 48q^{11} \\
&\quad + 96q^{12} + 60q^{13} + 84q^{14} \\
&\quad + 136q^{15} + 84q^{16}
\end{aligned}$$

$$\begin{aligned}
&+168q^{17} + 208q^{18} + 180q^{19} + 304q^{20} \\
&\quad + 272q^{21} + 232q^{22} \\
&\quad + 356q^{23} + 296q^{24} \\
&\quad + 356q^{25} + \dots,
\end{aligned}$$

$$\begin{aligned}
\Theta_{F_1 \oplus \Phi_3}(q) &= \Theta_{F_1}(q)\Theta_{\Phi_3}(q) \\
&= 1 + 2q + 6q^2 + 12q^3 \\
&\quad + 14q^{14} + 24q^5 + 20q^6 \\
&\quad + 16q^7 + 30q^8
\end{aligned}$$

$$\begin{aligned}
&+14q^9 + 40q^{10} + 60q^{11} + 36q^{12} + 78q^{13} \\
&\quad + 66q^{14} + 52q^{15} + 140q^{16}
\end{aligned}$$

$$\begin{aligned}
&+132q^{17} + 150q^{18} + 258q^{19} + 168q^{20} \\
&\quad + 240q^{21} + 328q^{22} \\
&\quad + 198q^{23} + 392q^{24} \\
&\quad + 366q^{25} + \dots,
\end{aligned}$$

$$\begin{aligned}
\Theta_{F_3 \oplus \Psi_1}(q) &= \Theta_{F_3}(q)\Theta_{\Psi_1}(q) \\
&= 1 + 6q + 12q^2 + 8q^3 \\
&\quad + 8q^4 + 36q^5 + 48q^6 \\
&\quad + 18q^7 \\
&+38q^8 + 114q^9 + 112q^{10} + 52q^{11} \\
&\quad + 92q^{12} + 180q^{13} + 146q^{14} \\
&\quad + 84q^{15} + 132q^{16}
\end{aligned}$$

$$\begin{aligned}
&+234q^{17} + 276q^{18} + 176q^{19} + 176q^{20} \\
&\quad + 348q^{21} + 312q^{22} \\
&\quad + 168q^{23} + 252q^{24} \\
&\quad + 426q^{25} + \dots,
\end{aligned}$$

$$\begin{aligned}
\Theta_{F_2 \oplus \Psi_2}(q) &= \Theta_{F_2}(q)\Theta_{\Psi_2}(q) \\
&= 1 + 4q + 4q^2 + 8q^4 \\
&\quad + 24q^5 + 16q^6 + 4q^7 \\
&\quad + 44q^8
\end{aligned}$$

$$\begin{aligned}
&+84q^9 + 40q^{10} + 24q^{11} + 120q^{12} \\
&\quad + 152q^{13} + 72q^{14} + 88q^{15} \\
&\quad + 220q^{16} + 236q^{17}
\end{aligned}$$

$$\begin{aligned}
&+156q^{18} + 176q^{19} + 256q^{20} + 320q^{21} \\
&\quad + 280q^{22} + 248q^{23} \\
&\quad + 400q^{24} + 500q^{25} + \dots,
\end{aligned}$$

$$\begin{aligned}
\Theta_{F_1 \oplus \Psi_3}(q) &= \Theta_{F_1}(q)\Theta_{\Psi_3}(q) \\
&= 1 + 2q + 8q^4 + 12q^5 \\
&\quad + 6q^7 + 42q^8 + 38q^9 \\
&\quad + 36q^{11}
\end{aligned}$$

$$\begin{aligned}
&+116q^{12} + 76q^{13} + 18q^{14} + 132q^{15} \\
&\quad + 216q^{16} + 126q^{17}
\end{aligned}$$

$$\begin{aligned}
&96q^{18} + 240q^{19} + 288q^{20} + 228q^{21} \\
&\quad + 280q^{22} + 336q^{23} \\
&\quad + 404q^{24} + 454q^{25} + \dots,
\end{aligned}$$

$$\begin{aligned}
\Theta_{\Phi_3 \oplus \Psi_1}(q) &= \Theta_{\Phi_3}(q)\Theta_{\Psi_1}(q) \\
&= 1 + 6q^2 + 14q^4 + 20q^6 \\
&\quad + 2q^7 + 32q^8 + 12q^9 \\
&\quad + 52q^{10}
\end{aligned}$$

$$\begin{aligned}
&+24q^{11} + 60q^{12} + 22q^{13} + 72q^{14} + 36q^{15} \\
&\quad + 116q^{16} + 86q^{17} + 174q^{18}
\end{aligned}$$

$$\begin{aligned}
&+114q^{19} + 232q^{20} + 144q^{21} + 292q^{22} \\
&\quad + 238q^{23} + 368q^{24} \\
&\quad + 336q^{25} + \dots,
\end{aligned}$$

$$\begin{aligned}
\Theta_{\Phi_2 \oplus \Psi_2}(q) &= \Theta_{\Phi_2}(q)\Theta_{\Psi_2}(q) \\
&= 1 + 4q^2 + 8q^4 + 16q^6 \\
&\quad + 4q^7 + 28q^8 + 16q^9 \\
&\quad + 40q^{10}
\end{aligned}$$

$$+24q^{11} + 56q^{12} + 36q^{13} + 76q^{14} + 64q^{15} \\ + 120q^{16} + 84q^{17} + 164q^{18}$$

$$+116q^{19} + 192q^{20} + 160q^{21} + 288q^{22} \\ + 252q^{23} + 376q^{24} \\ + 368q^{25} + \dots,$$

$$\Theta_{\Phi_1 \oplus \Psi_3}(q) = \Theta_{\Phi_1}(q) \Theta_{\Psi_3}(q) \\ = 1 + 2q^2 + 6q^4 + 12q^6 \\ + 6q^7 + 20q^8 + 12q^9 \\ + 36q^{10}$$

$$+24q^{11} + 44q^{12} + 50q^{13} + 84q^{14} + 60q^{15} \\ + 116q^{16} + 114q^{17} \\ + 146q^{18}$$

$$+110q^{19} + 232q^{20} + 200q^{21} + 252q^{22} \\ + 282q^{23} + 392q^{24} \\ + 328q^{25} + \dots,$$

$$\Theta_{F_2 \oplus \Phi_1 \oplus \Psi_1}(q) = \Theta_{F_2}(q) \cdot \Theta_{\Phi_1 \oplus \Psi_1}(q) \\ = 1 + 4q + 6q^2 + 8q^3 \\ + 14q^4 + 16q^5 + 20q^6 \\ + 34q^7$$

$$+40q^8 + 48q^9 + 68q^{10} + 80q^{11} + 76q^{12} \\ + 82q^{13} + 108q^{14} + 100q^{15} \\ + 120q^{16}$$

$$+182q^{17} + 190q^{18} + 214q^{19} + 240q^{20} \\ + 256q^{21} + 292q^{22} \\ + 314q^{23} + 336q^{24} \\ + 380q^{25} + \dots,$$

$$\Theta_{F_1 \oplus \Phi_2 \oplus \Psi_1}(q) = \Theta_{\Phi_2}(q) \cdot \Theta_{F_1 \oplus \Psi_1}(q) \\ = 1 + 2q + 4q^2 + 8q^3 + 8q^4 \\ + 12q^5 + 16q^6 + 18q^7$$

$$+30q^8 + 38q^9 + 48q^{10} + 52q^{11} + 60q^{12} \\ + 64q^{13} + 62q^{14} + 92q^{15} \\ + 104q^{16}$$

$$+130q^{17} + 172q^{18} + 196q^{19} + 208q^{20} \\ + 236q^{21} + 272q^{22} \\ + 292q^{23} + 372q^{24} \\ + 398q^{25} + \dots,$$

$$\Theta_{F_1 \oplus \Phi_1 \oplus \Psi_2}(q) = \Theta_{\Psi_2}(q) \cdot \Theta_{F_1 \oplus \Phi_1}(q) \\ = 1 + 2q + 2q^2 + 4q^3 + 6q^4 \\ + 8q^5 + 12q^6$$

$$+20q^7 + 26q^8 + 30q^9 + 48q^{10} + 52q^{11} \\ + 52q^{12} + 74q^{13} + 94q^{14} \\ + 100q^{15} + 116q^{16} \\ + 168q^{17}$$

$$+162q^{18} + 158q^{19} + 216q^{20} + 240q^{21} \\ + 264q^{22} + 314q^{23} \\ + 392q^{24} + 414q^{25} + \dots,$$

and the Eisenstein series

$$E(q; F_4) = \dots = E(q; F_1 \oplus \Phi_1 \oplus \Psi_2) \\ = 1 + \frac{24}{1061} q + \frac{24 \cdot 9}{1061} q^2$$

$$+ \frac{24 \cdot 28}{1061} q^3 + \frac{24 \cdot 73}{1061} q^4 + \frac{24 \cdot 126}{1061} q^5 \\ + \frac{24 \cdot 252}{1061} q^6$$

$$+ \frac{24 \cdot 344}{1061} q^7 + \frac{24 \cdot 585}{1061} q^8 + \frac{24 \cdot 757}{1061} q^9 \\ + \frac{24 \cdot 1134}{1061} q^{10} \\ + \frac{24 \cdot 1332}{1061} q^{11}$$

$$+ \frac{24 \cdot 2044}{1061} q^{12} + \frac{24 \cdot 2198}{1061} q^{13} \\ + \frac{24 \cdot 3096}{1061} q^{14} \\ + \frac{24 \cdot 3528}{1061} q^{15} \\ + \frac{24 \cdot 4681}{1061} q^{16}$$

$$\begin{aligned}
& + \frac{24 \cdot 4914}{1061} q^{17} + \frac{24 \cdot 6813}{1061} q^{18} \\
& \quad + \frac{24 \cdot 6860}{1061} q^{19} \\
& \quad + \frac{24 \cdot 9198}{1061} q^{20} \\
& \quad + \frac{24 \cdot 9632}{1061} q^{21} \\
& + \frac{24 \cdot 11988}{1061} q^{22} + \frac{24 \cdot 12168}{1061} q^{23} \\
& \quad + \frac{24 \cdot 16380}{1061} q^{24} \\
& \quad + \frac{24 \cdot 15751}{1061} q^{25} + \dots
\end{aligned}$$

are linear combinations of the 20 generalized theta series mentioned in Theorem [1]. The coefficients are given in TABLE_of_COEFFICIENTS103 in[9].

Proof. Let's see the situation in the case:

$$\begin{aligned}
\Theta_{F_4}(q) - E(q; F_4) &= c_1 \Theta_{F_2, \varphi_{11}}(q) \\
&+ c_2 \Theta_{\Phi_2, \varphi_{11}}(q) \\
&+ c_3 \Theta_{\Phi_2, \varphi_{12}}(q) + \\
&c_4 \Theta_{F_1 \oplus \Phi_1, \varphi_{11}}(q) + c_5 \Theta_{F_1 \oplus \Phi_1, \varphi_{12}}(q) \\
&+ c_6 \Theta_{F_1 \oplus \Phi_1, \varphi_{33}}(q) \\
&+ c_7 \Theta_{F_1 \oplus \Phi_1, \varphi_{34}}(q) \\
&+ c_8 \Theta_{\Psi_2, \varphi_{11}}(q) + c_9 \Theta_{\Psi_2, \varphi_{12}}(q) \\
&+ c_{10} \Theta_{F_1 \oplus \Psi_1, \varphi_{11}}(q) \\
&+ c_{11} \Theta_{F_1 \oplus \Psi_1, \varphi_{12}}(q) \\
&+ c_{12} \Theta_{F_1 \oplus \Psi_1, \varphi_{33}}(q) + c_{13} \Theta_{F_1 \oplus \Psi_1, \varphi_{34}}(q) \\
&+ c_{14} \Theta_{\Phi_1 \oplus \Psi_1, \varphi_{11}}(q) \\
&+ c_{15} \Theta_{\Phi_1 \oplus \Psi_1, \varphi_{12}}(q) \\
&+ c_{16} \Theta_{\Phi_1 \oplus \Psi_1, \varphi_{22}}(q) \\
&+ c_{17} \Theta_{\Phi_1 \oplus \Psi_1, \varphi_{23}}(q) + c_{18} \Theta_{\Phi_1 \oplus \Psi_1, \varphi_{33}}(q) \\
&+ c_{19} \Theta_{\Phi_1 \oplus \Psi_1, \varphi_{34}}(q) \\
&+ c_{20} \Theta_{\Phi_1 \oplus \Psi_1, \varphi_{44}}(q).
\end{aligned}$$

By equating the coefficients of q^n in both sides for $n = 1, 2, 3, \dots, 25$, we see that there

exist solutions of the equations in coefficients. We repeat the same procedure for the other cases. At the end, by solving 25 linear equations in 20 variables we get all the coefficients. We also observe that c_{20} is always zero.

Conclusion

The representation numbers for the quadratic forms

$$\begin{aligned}
H = F_4, \Phi_4, \Phi_3 \oplus \Phi'_1, \Phi_2 \oplus \Phi'_2, \Phi_1 \\
\oplus \Phi'_3, \Phi'_4, \Psi_4, \Psi_3 \oplus \Psi'_1, \Psi_2 \\
\oplus \Psi'_2, \Psi_1 \oplus \Psi'_3, \Psi'_4,
\end{aligned}$$

$$\begin{aligned}
F_3 \oplus \Phi_1, F_3 \oplus \Phi'_1, F_2 \oplus \Phi_2, F_2 \oplus \Phi_1 \\
\oplus \Phi'_1, F_2 \oplus \Phi'_2, F_1 \oplus \Phi_3, F_1 \\
\oplus \Phi_2 \oplus \Phi'_1,
\end{aligned}$$

$$\begin{aligned}
F_1 \oplus \Phi_1 \oplus \Phi'_2, F_1 \oplus \Phi'_3, F_3 \oplus \Psi_1, F_3 \\
\oplus \Psi'_1, F_2 \oplus \Psi_2, F_2 \oplus \Psi_1 \\
\oplus \Psi'_1, F_2 \oplus \Psi'_2, F_1 \oplus \Psi_3,
\end{aligned}$$

$$\begin{aligned}
F_1 \oplus \Psi_2 \oplus \Psi'_1, F_1 \oplus \Psi_1 \oplus \Psi'_2, F_1 \oplus \Psi'_3, \Phi_3 \\
\oplus \Psi_1, \Phi_2 \oplus \Phi'_1 \oplus \Psi_1, \Phi_1 \\
\oplus \Phi'_2 \oplus \Psi_1,
\end{aligned}$$

$$\begin{aligned}
\Phi'_3 \oplus \Psi_1, \Phi_3 \oplus \Psi'_1, \Phi_2 \oplus \Phi'_1 \oplus \Psi'_1, \Phi_1 \\
\oplus \Phi'_2 \oplus \Psi'_1, \Phi'_3 \oplus \Psi'_1, \Phi_2 \\
\oplus \Psi_2, \Phi'_2 \oplus \Psi_2,
\end{aligned}$$

$$\begin{aligned}
\Phi_2 \oplus \Psi'_2, \Phi'_2 \oplus \Psi'_2, \Phi_1 \oplus \Phi'_1 \oplus \Psi_2, \Phi_1 \\
\oplus \Phi'_1 \oplus \Psi_1 \oplus \Psi'_1, \Phi_1 \oplus \Phi'_1 \\
\oplus \Psi'_2, \Phi_2 \oplus \Psi_1 \oplus \Psi'_1, \\
\Phi'_2 \oplus \Psi_1 \oplus \Psi'_1, \Psi_3 \oplus \Phi_1, \Psi_2 \oplus \Psi'_1 \\
\oplus \Phi_1, \Psi_1 \oplus \Psi'_2 \oplus \Phi_1, \Psi'_3 \\
\oplus \Phi_1, \Psi_3 \oplus \Phi'_1, \Psi_2 \oplus \Psi'_1 \\
\oplus \Phi'_1,
\end{aligned}$$

$$\begin{aligned}
\Psi_1 \oplus \Psi'_2 \oplus \Phi'_1, \Psi'_3 \oplus \Phi'_1, F_2 \oplus \Phi_1 \\
\oplus \Psi_1, F_2 \oplus \Phi'_1 \oplus \Psi_1, F_2 \\
\oplus \Phi_1 \oplus \Psi'_1, F_2 \oplus \Phi'_1 \oplus \Psi'_1,
\end{aligned}$$

$F_1 \oplus \Phi_2 \oplus \Psi_1, F_1 \oplus \Phi'_2 \oplus \Psi_1, F_1 \oplus \Phi_2$
 $\oplus \Psi'_1, F_1 \oplus \Phi'_2 \oplus \Psi'_1, F_1$
 $\oplus \Phi_1 \oplus \Phi'_1 \oplus \Psi_1,$
 $F_1 \oplus \Phi_1 \oplus \Phi'_1 \oplus \Psi'_1, F_1 \oplus \Phi_1 \oplus \Psi_2, F_1$
 $\oplus \Phi'_1 \oplus \Psi_2, F_1 \oplus \Phi_1$
 $\oplus \Psi'_2, F_1 \oplus \Phi'_1 \oplus \Psi'_2,$
 $F_1 \oplus \Phi_1 \oplus \Psi_1 \oplus \Psi'_1, F_1 \oplus \Phi'_1 \oplus \Psi_1 \oplus \Psi'_1,$
 are

$$\begin{aligned}
 r(n; H) &= \frac{24}{1061} \sigma_3^*(n) \\
 &+ \frac{c_1}{103} \sum_{F_2=n} (103x_1^2 - 26n) \\
 &+ \frac{c_2}{103} \sum_{\Phi_2=n} (103x_1^2 - 13n) \\
 &+ \frac{c_3}{2 \cdot 103} \sum_{\Phi_2=n} (2 \cdot 103x_1x_2 + n) \\
 &+ \frac{c_4}{103} \sum_{F_1 \oplus \Phi_1=n} (103x_1^2 - 26n) \\
 &+ \frac{c_5}{2 \cdot 103} \sum_{F_1 \oplus \Phi_1=n} (2 \cdot 103x_1x_2 + n) \\
 &+ \frac{c_6}{103} \sum_{F_1 \oplus \Phi_1=n} (103x_3^2 - 13n) \\
 &+ \frac{c_7}{2 \cdot 103} \sum_{F_1 \oplus \Phi_1=n} (2 \cdot 103x_3x_4 + n) \\
 &+ \frac{c_8}{103} \sum_{\Psi_2=n} (103x_1^2 - 7n) \\
 &+ \frac{c_9}{2 \cdot 103} \sum_{\Psi_2=n} (2 \cdot 103x_1x_2 + n) \\
 &+ \frac{c_{10}}{103} \sum_{F_1 \oplus \Psi_1=n} (103x_1^2 - 26n)
 \end{aligned}$$

$$\begin{aligned}
 &+ \frac{c_{11}}{2 \cdot 103} \sum_{F_1 \oplus \Psi_1=n} (2 \cdot 103x_1x_2 + n) \\
 &+ \frac{c_{12}}{103} \sum_{F_1 \oplus \Psi_1=n} (103x_3^2 - 7n) \\
 &+ \frac{c_{13}}{206} \sum_{F_1 \oplus \Psi_1=n} (206x_3x_4 + n) \\
 &+ \frac{c_{14}}{103} \sum_{\Phi_1 \oplus \Psi_1=n} (103x_1^2 - 13n) \\
 &+ \frac{c_{15}}{2 \cdot 103} \sum_{\Phi_1 \oplus \Psi_1=n} (2 \cdot 103x_1x_2 + n) \\
 &+ \frac{c_{16}}{103^2} \sum_{\Phi_1 \oplus \Psi_1=n} (103^2x_2^2 - 182n) \\
 &+ \frac{c_{17}}{103^2} \sum_{\Phi_1 \oplus \Psi_1=n} (103^2x_2x_3 + 28n) \\
 &+ \frac{c_{18}}{103} \sum_{\Phi_1 \oplus \Psi_1=n} (103x_3^2 - 7n) \\
 &+ \frac{c_{19}}{2 \cdot 103} \sum_{\Phi_1 \oplus \Psi_1=n} (2 \cdot 103x_3x_4 + n)
 \end{aligned}$$

the coefficients $\begin{pmatrix} c_1 = \\ c_2 = \\ c_3 = \\ c_4 = \\ c_5 = \\ c_6 = \\ c_7 = \\ c_8 = \\ c_9 = \\ c_{10} = \\ c_{11} = \\ c_{12} = \\ c_{13} = \\ c_{14} = \\ c_{15} = \\ c_{16} = \\ c_{17} = \\ c_{18} = \\ c_{19} = \end{pmatrix}$ corresponding to the

quadratic form H are given in the TABLE_of_COEFFICIENTS103 in [9]

Proof. It follows from the preceding Theorem.

Conclusion

$S_4(\Gamma_0(103))$'s basis of a subspace has been found via generalized theta series associated to the direct sums of the quadratic forms $x_1^2 + x_1x_2 + 26x_2^2$, $2x_1^2 \pm x_1x_2 + 13x_2^2$, $4x_1^2 \pm 3x_1x_2 + 7x_2^2$.

REFERENCES

[1]B. Kendirli, "Evaluation of Some Convolution Sums by Quasimodular Forms", European Journal of Pure and Applied Mathematics ISSN 13075543 Vol.8., No. 1, Jan. 2015, pp. 81-110.

[2]B. Kendirli, "Evaluation of Some Convolution Sums and Representation Numbers of Quadratic Forms of Discriminant 135", British Journal of Mathematics and Computer Science, Vol6/6, Jan. 2015, pp. 494-531.

[3]B. Kendirli, *Evaluation of Some Convolution Sums and the Representation numbers*, ArsCombinatorica Volume CXVI, July, pp 65-91.

[4]B. Kendirli, *Cusp Forms in $S_4(\Gamma_0(79))$ and the number of representations of positive integers by some direct sum of binary quadratic forms with discriminant -79* , Bulletin of the Korean Mathematical Society Vol 49/3 2012.

[5]B. Kendirli, *Cusp Forms in $S_4(\Gamma_0(47))$ and the number of representations of positive integers by some direct sum of binary quadratic forms with discriminant -47* , Hindawi, International Journal of Mathematics and Mathematical Sciences Vol 2012, 303492 10 pages.

[6]B. Kendirli, *The Bases of $M_4(\Gamma_0(71)), M_6(\Gamma_0(71))$ and the Number of Representations of Integers*, Hindawi, Mathematical Problems in Engineering Vol 2013, 695265, 34 pages.

[7]Ivaniec H., Kowalski,E,Analytic Number Theory, American Mathematical Society Colloquium publications volume 53, 2000.

[8]T.Miyake, *Modular Forms* Springer-Verlag 1989.

[9]<http://www.bariskendirli.com.tr>