

Cost Estimation Comparisons Between Least Square Regression and Quantile Regression on Fiber Reinforced Bridge Projects

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Abstract

This paper focuses on the Least Square (LS) regression using the mean and Quantile (M) regression analysis using median which is based on “well-Known” parametric estimation methodologies. Data from Oregon and California highway bridges were used for the comparison of the two methods. Relationships were developed to predict the unit cost of FRP repair work and FRP cost was found to have a high degree of correlation with FRP area for both Oregon and California. It was observed that the Cost Estimating Relationships (CERs) obtained by Quantile (M) regression method had the smaller Mean Absolute Deviation (MAD) values and lower Mean Absolute Percentage Error (MAPE) values than Least Square (LS) regression. The study showed that Quantile Regression is much less sensitive to outliers than Least Squares Regression.

Keywords: *Quantile regression, cost estimating relationships (cers), fiber reinforced polymer (frp) wraps, outliers*

1. Introduction

State transportation departments are faced with a challenge to keep bridges under their jurisdiction in good operating conditions. Corrosion of bridges has been a constant challenge for engineers and new materials (e.g. polymers, metals, ceramics and their composites) are being developed to minimize corrosion related issues. Fiber Reinforced Polymers (FRPs) are composites that combine the strength of fibers with the stability of polymer resins. The strength of FRP materials comes from the type of fibers used, usually they are glass, carbon or aramid fibers. The primary advantages of FRP materials are: lightweight, non-magnetic, high corrosion resistance and high strength to weight ratio. These advantages make FRP materials a viable option for the initial construction or for

rehabilitation of current bridges. As a new technology, it is hampered by a lack of approved standards and an effective cost estimation methodology.

Cost Estimating Relationships (CERs) are the parametric equations that are developed between different variables to determine the cost of a project component or the total cost of the project. These relationships are developed by cost estimators and are used by managers to control the variables that impact the cost of a project [4]. The primary statistical method to develop CERs is Least Square (LS) regression which is based upon the mean of the data. An alternative method is the Quantile (Q) regression which is based upon the median of the data [6]. This project compares

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the two methods using limited data on highway rehabilitation projects in Oregon and California utilizing the software ESTIMFEC [1] to perform the analysis. One of the problems of using historical data is that it may contain outliers and Quantile regression is less sensitive to outliers than the Least Square regression.

2. Methodology

Least Square regression is the most commonly used method to develop linear predictive relationships including cost relationships and engineering relationships. Quantile methods are commonly used to determine percentile rankings used on many testing scores and in areas such as finance and sociology. Quantile regression methods have been used within labor or educational economies to study wage determinants, trends in income inequality or effects of socioeconomic characteristics on education attainment [3].

2.1 Least Square (LS) Regression

Least Square regression is a statistical method of analysis that estimates the relationship between one or more independent variables and the dependent variable by minimizing the sum of squares of the difference between the observed and predicted values of the dependent variable [8]. A simple Least Square regression model involving only one independent variable (X) predicting a dependent variable (Y) is expressed by Equation 1.

$$Y = a + bX + \varepsilon \quad (1)$$

Here “a” is the intercept that indicates where the straight line intersects Y-axis; “b” is the slope that indicates the degree of steepness of the straight line and “ε” represents the error. The best line or a relationship would be the one with the Least Sum of Squared Errors (SSE) [9].

$$SSE = \sum(Y_i - \hat{Y})^2 \quad (2)$$

Where

Y_i = Dependent variable i and where $i = 1, \dots, n$

$$\hat{Y} = (\sum Y_i) / n \quad (3)$$

2.1.1 Drawbacks of Least Square (LS) Regression

According to Foussier [4], the main drawback is “regression”. If there are a large number of observations and they are scattered, the Least Square method underestimates the larger observations and overestimates the smaller observations as the Least Square method regresses towards the average value. It is a major drawback, since the costs of a product or project are always scattered, and there is no necessity for the cost to regress towards the average cost. The Least Squares regression is based upon the data having a normal distribution and this does not always occur. If the data is truly normal, then the mean and median would be the same. The other major disadvantage of the Least Square method is that it is sensitive to outlier that is due to squaring of the error term. Since the Least Square method is not robust, this might have a tremendous impact on the predicted cost. Foussier [4] indicates it is due to the small break down of the arithmetic average. Hence, the change of just one data point might have a severe impact on the predicted cost. The outliers that are far away from the tentative CER will strongly impact the predicted cost and the R^2 . A tendency to eliminate the outliers to increase R^2 might cause one to eliminate too many data points.

According to Xia [10], the third disadvantage of Least Square method is multi-collinearity. There may be several variables that may be linearly collinear to each other in the data. It may adversely affect the coefficient of the estimates when there is a small change in the data. The relationship thus obtained between these variables does not make sense even though it is mathematically correct and produces a large confidence interval. This is due to the fact that to estimate the values of the parameters one needs to compute the inverse of the matrix based on the inputs. The determinant of a matrix with two collinear variables is equal to zero and hence these matrices cannot be inverted reliably to estimate the parameters [4]. However the relationships developed in this study were concerned with only one variable.

2.2 Quantile Regression

According to Koenker [6] Quantile Regression is the estimation of Quantiles of the conditional distribution. It models the relation between a set of predictor variables and the specific Quantiles of the response variable. The prominent form of Quantile regression is the Median regression. Koenker and Bassett stated that for a sample Y_1, Y_2, \dots, Y_n from a population Y , we can compute the sample median ν by minimizing the sum of absolute deviations.

$$\hat{\nu} = \min \sum_{i=1}^n |Y_i - \nu| \tag{4}$$

Similarly we can compute the τ th sample Quantile $\nu(\tau)$ by minimizing the absolute deviations of various Quantiles.

$$\hat{\nu}(\tau) = \min \sum_{i=1}^n \rho_{\tau} |Y_i - \nu| \tag{5}$$

Here $\rho_{\tau}(z) = z(\tau - I(z < 0)), 0 < \tau < 1$. $I(\cdot)$ denotes the Indicator function. Koenker stated that the optimal value of τ is the median. The estimate of linear conditional Quantile function (i.e. of $Q_{\tau}(\frac{Y_i}{X_i}) = X_i\beta_{\tau}$) can be found by the criterion:

$$\hat{\beta}(\tau) = \arg \min \sum_{i=1}^n \rho_{\tau} |Y_i - X_i\beta| \tag{6}$$

Where $\hat{\beta}(\tau)$ is called the τ th regression Quantile. The minimizer function $\sum_{i=1}^n \rho_{\tau} |Y_i - X_i\beta|$ is differentiable except at $Y_i = X_i\beta$. A simplex based algorithm was developed by Barrodale and Roberts [2] to solve the minimizer function and was later extended by Koenker and d'Orey [7] to Quantile regression estimation. The case where $\tau = 1/2$ corresponds to median regression, which is also known as L1 regression.

3. Data Source and Analysis

3.1 Oregon Bridges

The state of Oregon had several bridges that have undergone repair and rehabilitation work in recent times. There were seventeen bridges that had undergone repair work using FRP wrappings for which the data was obtained. The area of the bridges ranges from 500 ft² to 9,204 ft². There were multiple bids that were quoted for each bridge repair work by several contractors. Table 3.1 gives the total contract cost, FRP area, FRP cost and other calculated parameters.

Table 3.1: Summary of Year, Accepted FRP Cost, Accepted FRP Unit Cost and Total Contract of Oregon Highway Bridges

Location	Year	Bridge No.	FRP Area, ft ²	Accepted FRP Unit Cost, \$/ft ²	Accepted FRP Cost, \$	Total Contract, \$	% of FRP Cost of Total Contract
OR	2007	01786A	3,144	50.68	159,343	12,537,779	1.27
	2008	07601B	712	40.82	29,062	2,548,552	1.14
	2008	08381S	1,280	39.30	50,299	2,548,552	1.97
	2008	7458	1,955	97.20	190,020	10,193,994	1.86
	2008	08383N	6,103	27.47	167,665	3,324,179	5.04
	2009	8843	2,345	106.82	250,480	3,442,437	7.28
	2009	03173A	9,204	13.48	124,069	3,885,033	3.19
	2010	7530	500	61.11	30,557	2,002,807	1.53
	2010	7532	500	61.11	30,557	2,002,807	1.53
	2010	7533	500	13.10	6,548	2,002,807	0.33
	2010	7534	500	61.11	30,557	2,002,807	1.53
	2010	06945A	840	38.98	32,739	3,013,112	1.09
	2010	04981A	1,900	31.59	60,022	976,687	6.15
	2010	02233A	1,900	40.20	76,391	976,687	7.82
	2010	04979A	2,600	27.70	72,026	976,687	7.37
	2010	02236A	3,200	27.28	87,304	976,687	8.94
	2010	7392	5,800	30.11	174,609	457,704	38.15

Note: All the cost data has been adjusted for inflation to 2013 dollar value

The accepted FRP cost and FRP area are the two variables that are used to predict the unit cost of FRP repair work. The total FRP cost constitutes a minor fraction of the total contract cost. Hence during the estimation of unit cost of FRP repair work, FRP cost is used instead of the total contract cost. Only one layer of FRP application was considered due to limited availability of the data. Bridges with accepted FRP cost of less than 1% of the total contract cost were omitted from the paper. Bridge No. 7533 was found to have a value of 0.33% (Table 3.1) and was not considered in the study as its cost was more than 70 percent lower than three other bridges with the exact same area. Figure 3.1 shows the scatter plot of FRP cost and FRP area. Figure 3.1 indicates three points (i.e. bridges 7458, 8843, 03173A) to be potential outliers. The R^2 value obtained for all the data points included is 0.278 and coefficient of determination (R^*) value obtained for Quantile (Median) regression is 0.430. The bridge 01786A was not considered to be an outlier since its FRP cost was within 95% of the confidence interval and no physical characteristic was detected that indicated it to be an outlier.

It can be observed from Figure 3.1 that the Least Square regression method overestimated smaller values and underestimated larger values when compared to Quantile (Median) regression. The slope of the two equations are different and the Least Squares is greatly affected by the outliers as indicated by Equations 7 and

8. The regression equations obtained by Least Square and Quantile (Median) methods for all the data included are:

$$\text{Accepted FRP Cost (LS-\$)} = 57,515 + 15.19 * \text{FRP Area (ft}^2\text{)} \quad (7)$$

$$\text{Accepted FRP Cost (Q (M)-\$)} = 18,321 + 24.47 * \text{FRP Area (ft}^2\text{)} \quad (8)$$

The coefficient of determination values obtained are 0.278 (LS, R^2) and 0.430 (Q (M), R^*) respectively. The equations have very different slopes and some of the data points appear as outliers. When the outliers are detected and removed, the two methods will be in more agreement.

Further analysis for outliers is made by conducting a Tietjen-Moore Test for multiple outlier detections. The number of suspected outliers in the test were considered to be three (i.e. $k = 3$). The value of test static E^k to be 0.2131 which is less than the critical value of 0.2249 thus confirming the presence of three outliers in the data. When the number of suspected outliers in the test were considered to be four (i.e. $k = 4$) the test static (E^k) value (0.1910) was greater than the critical value (0.1508) and did not confirm four outliers. Further residual analysis was carried out to identify these outliers.

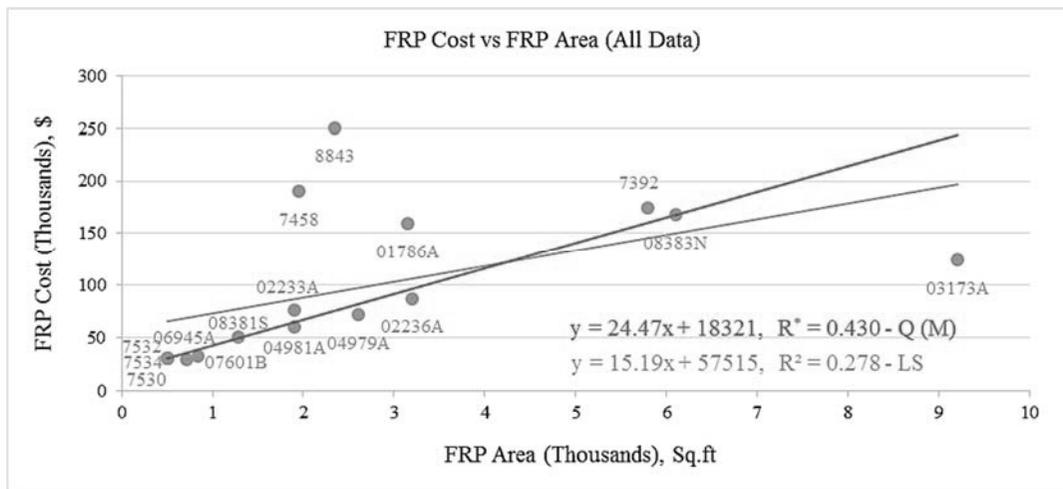


Figure. 3.1: Scatter Plot (All Data) of Accepted FRP Cost and FRP Area of Oregon Highway Bridges (Q (M) & LS) with $R^2 = 0.278$ & $R^* = 0.430$

The computations to identify the outliers are made in three steps:

1. A linear relationship is obtained using all the data points and the prediction of FRP cost for each bridge is made using this relationship. Deviations of the predicted FRP cost from the actual FRP cost is computed as “Deviations 1”.
2. Then each data point is successively discarded and therefore N new relationships are computed. Appendix 1 shows the R² corresponding to the new relationship without this data point and the deviation, called “Deviation 2” between this data point and the new relationship. A few R² are much higher (>0.100) than the corresponding data points are identified as outliers since their deletion greatly improves the relationship.
3. The difference between both the deviations is divided by the standard deviation of “Deviation 1” and is multiplied by a factor of 100. The column “Relative Variation” from Appendix 1 gives the resultant value. The data points that have a relative variation of more than ±50% are considered to be outliers in this paper.

Appendix 1 shows that bridges 7458 and 03173A had a relative variation more than the threshold value of ±50% thus indicating them to be possible outliers. Appendix 2 also shows that removing the bridge 8843 greatly improves the correlation between the FRP cost and FRP area.

- Bridge 03173A is a seismic retrofit involving different repair procedure than the rest of the bridges thus having a much higher cost than the rest of the bridges.
- Bridge 7458 involved construction and removal of various temporary work platforms with enclosure. The humidity and temperature in the enclosure was maintained using an HVAC system which is the likely reason for higher unit cost of FRP repair work.
- Bridge 8843 has many elements that contain asbestos. The contractor had to remove these

materials before proceeding with the FRP application.

Bridges 03173A, 7458 and 8843 were identified as spurious bids considering their unusual FRP repair cost. These bridges were considered as outliers and were not considered in further analysis of the data.

Equation 9 was obtained for accepted FRP cost against FRP area for all the bridges in Oregon by Least Square method with an R² of 0.884 is given by:

$$\text{Accepted FRP Cost (LS-\$)} = 16,736 + 27.03 * \text{FRP Area (ft}^2\text{)} \quad (9)$$

Equation 10 was obtained by performing Quantile (Median) regression for accepted FRP cost against the FRP repair area for all the bridges in Oregon with an R* value of 0.773 is given by:

$$\text{Accepted FRP Cost (Q (M)-\$)} = 18,321 + 24.47 * \text{FRP Area (ft}^2\text{)} \quad (10)$$

The intercept values (16,736 (LS), 18,321 (Q (M))) represent the fixed set-up costs such as mobilization, equipment and traffic control.

Figure 3.2 is developed between the accepted FRP cost against FRP area to compare the two methods (LS & Q (M)) of analysis. It shows the extreme observation point (01786A) had minimal impact on the relationship obtained by Quantile (Median) regression method. The slope difference (absolute) between the CERs of Figure 3.1 and Figure 3.2 is 0 (Q (M)) and 11.84 (LS) which indicates the robustness of Q (M) regression with respect to outliers. The slope of CER obtained by Least Square method changed more than 75% indicating a high influence of outlier on the Least Square method whereas the outliers has no effect on the CER obtained by the Quantile method. When the outliers are removed, agreement is high as indicated by Figure 3.2.

The results from Table 3.2 show that Least Square regression method has the least SSE value and Quantile (Median) regression method has the least MAD and MAPE values. Profits and costs are indicated better by the mean absolute deviation and by the mean absolute

percentage error than by the sum of squares of the errors. A relationship between the accepted FRP cost and the FRP area to predict the unit cost (\$/ft²) values, can be established. The predicted unit cost obtained from Equations 11 and 12 are given by:

$$\text{Unit Cost (LS-$/ft}^2\text{)} = 27.03 + 16,736/\text{FRP Area (ft}^2\text{)} \quad (11)$$

$$\text{Unit Cost (Q (M)-$/ft}^2\text{)} = 24.47 + 18,321/\text{FRP Area (ft}^2\text{)} \quad (12)$$

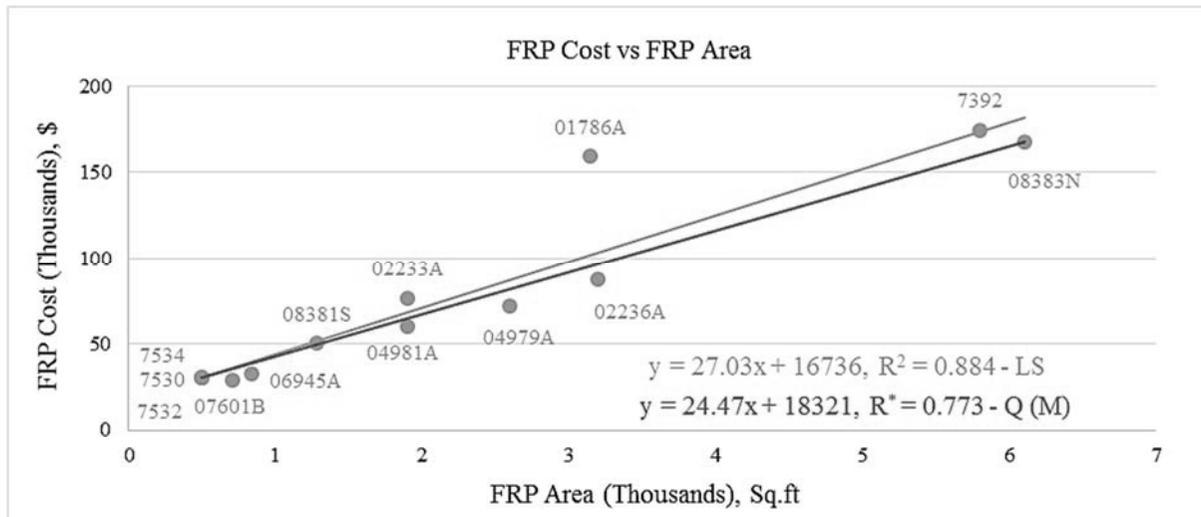


Figure. 3.2: Scatter Plot of Accepted FRP Cost and FRP Area of Oregon Highway Bridges (LS & Q (M)) with $R^2 = 0.884$ & $R^* = 0.773$

Table 3.3 shows the predicted FRP unit cost (\$/ft²) values obtained by the two methods (LS & Q (M)) of analysis using Equations 11 and 12.. These equations are similar as the outliers have been determined and removed, and thus agreement between the two methods is high. The actual unit costs versus predicted unit costs have good agreement overall, but the predicted equations are approximate only. There is the assumption that the start-up costs, which is indicated by the intercept of the curve, is constant for all project sizes. When the bridges are grouped into two ranges as

done in Table 3.3, the agreement is much better for the unit costs.

Table 3.4 gives the weighted average by area of FRP unit cost for the two different methods of analysis for two separate area ranges (i.e. <2000 & >2000). The weighted average values for both the Quantile method and the Least Squares method are near the actual unit cost values. The difference in values between the two area ranges indicate the large impact of area upon FRP unit cost

Table 3.2: Residual Analysis Table for Oregon Highway Bridges

Area, ft ²	Accepted FRP Cost, \$	Predicted FRP Cost, \$		SSE (X-Xp) ²	
		Least Square	Quantile (Median)	Least Square	Quantile (Median)
712	29,062	35,989	35,744	47,981,936	44,646,310
1,280	50,299	51,347	49,643	1,097,351	431,329
6,103	167,665	181,756	167,661	198,569,710	10
500	30,557	30,256	30,556	90,024	0
500	30,557	30,256	30,556	90,024	0

500	30,557	30,256	30,556	90,024	0
840	32,739	39,450	38,876	45,032,308	37,658,713
1,900	60,022	68,111	64,814	65,437,468	22,965,764
1,900	76,391	68,111	64,814	68,562,082	134,033,976
3,144	159,343	101,748	95,255	3,317,230,332	4,107,312,760
2,600	72,026	87,038	81,943	225,369,002	98,345,164
3,200	87,304	103,262	96,625	254,639,705	86,874,557
5,800	174,609	173,563	160,247	1,093,099	206,258,302
Total	1,001,130	1,001,144	947,289	4,225,283,064	4,738,526,887
MAD			Average	MAPE, %	
Area (ft²)	Least Square	Quantile (Median)		Least Square	Quantile (Median)
712	6,927	6,682		23.84	22.99
1280	1,048	657		2.08	1.31
6103	14,091	3		8.4	0
500	300	1		0.98	0
500	300	1		0.98	0
500	300	1		0.98	0
840	6,711	6,137		20.50	18.74
1,900	8,089	4,792		13.48	7.98
1,900	8,280	11,577		10.84	15.16
3,144	57,595	64,088		36.15	40.22
2,600	15,012	9,917		20.84	13.77
3,200	15,957	9,321		18.28	10.68
5,800	1,046	14,362		0.60	8.23
Average	10,435	9,811		12.15	10.70

Table 3.3: Summary of FRP Repair Unit Cost (\$/Ft²) Obtained For Oregon Highway Bridges

FRP Area, ft²	Actual FRP Cost, \$	Actual FRP Unit Cost, \$/ft²	Pred. FRP Unit Cost (LS), \$/ft²	Pred. FRP Unit Cost (Q (M)), \$/ft²
712	29,062	40.82	50.55	50.20
1,280	50,299	39.30	40.11	38.78
6,103	167,665	27.47	29.78	27.47
500	30,557	61.11	60.51	61.11
500	30,557	61.11	60.51	61.11
500	30,557	61.11	60.51	61.11
840	32,739	38.98	46.96	46.28
1,900	60,022	31.59	35.85	34.11
1,900	76,391	40.21	35.85	34.11
3,144	159,343	50.68	32.36	30.30
2,600	72,026	27.70	33.48	31.52
3,200	87,304	27.28	32.27	30.20
5,800	174,609	30.10	29.92	27.63

Table 3.4: Summary of Average FRP Unit Cost (\$/Ft²) Obtained For Respective Area Range of Oregon Bridges

FRP Area Range, ft ²	Average* FRP Unit Cost, \$/ft ² (LS)	Average* FRP Unit Cost, \$/ft ² (Q (M))	Average* FRP Unit Cost, \$/ft ² (Actual)
<2000 (8)	43.50	42.49	41.83
>2000 (5)	31.05	28.87	31.70

*The area of the respective bridges was considered as weight in calculating the average FRP unit cost

3.2 California

There were a total of seven bridges (Table 3.5) for which the contract data was obtained which had undergone FRP repair and rehabilitation work in California. These bridges were spread across Alameda County, Santa Barbara County, Tulare County, Los

Angeles County, San Bernardino County, Riverside County and Imperial County. There were multiple contract bids for each bridge. Table 3.5 gives the Contract number, FRP cost, FRP area, Total contract cost and other calculated parameters.

Table 3.5: Summary of Year, Accepted FRP Cost and Accepted Unit Cost for California Highway Bridges

Location	Contract No.	Date	Area, ft ²	Accepted Unit Cost, \$/ft ²	Accepted FRP Cost, \$	Total Contract, \$	%FRP Cost
CA	06-0N8504	2012	170	98.61	16,763	735,961	2.28%
	11-264004	2010	420	84.40	35,449	995,644	3.56%
	08-472304	2008	614	62.01	38,075	39,726,300	0.10%
	10-0W0704	2012	2,620	70.81	183,974	2,559,279	7.19%
	08-0G4804	2011	3,375	95.00	320,625	1,482,509	21.63%
	06-0C1304	2012	3,677	65.29	238,053	3,352,320	7.10%
	08-0M94U4	2012	4163	60.12	248,157	24,155,220	1.03%

Note: All the data has been adjusted for inflation to 2013 dollar values

The California bridge data (Table 3.5) was mainly categorized into FRP repair work. FRP area is the variable which was found to have a significant correlation with the total FRP cost. The regression equations obtained by Least Square and Quantile (Median) methods for all the data included are:

$$\text{Accepted FRP Cost (LS-\$)} = 7,202 + 68.53 * \text{FRP Area(ft}^2\text{)} \quad (13)$$

$$\text{Accepted FRP Cost (Q (M)-\$)} = 6,036 + 63.10 * \text{FRP Area(ft}^2\text{)} \quad (14)$$

The coefficient of determination values obtained are 0.898 (LS, R²) and 0.806 (Q (M), R^{*}) respectively.

Scatter plot (Figure 3.3) is plotted between the accepted FRP cost and FRP area for all the bridges in California.

Bridges with accepted FRP cost of less than 1% of the total contract cost are omitted in this paper. Contract No. 08-472304 (Table 3.5) was found to have value of 0.10% and was not considered in the study to be consistent with the Oregon study, but it did not appear to be an outlier in the California data. Contract No. 08-0G4804 is a seismic retrofit, which had a different repair procedure than the rest of the bridges and much higher unit costs. Hence it was not considered in the analysis. The regression equations are computed again without these two bridges.

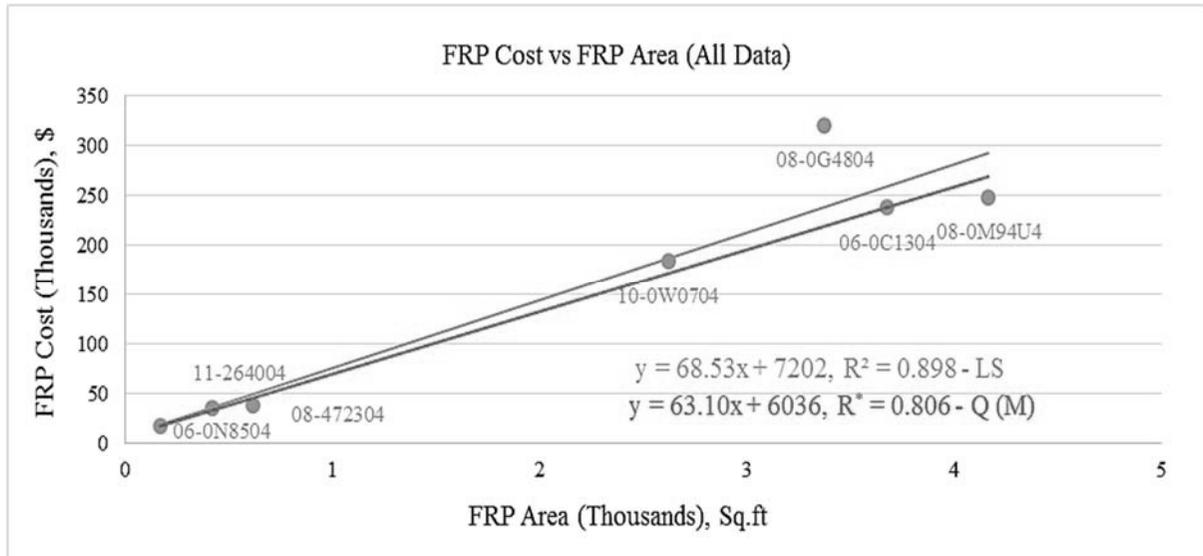


Figure. 3.3: Scatter Plot (All Data) of Accepted FRP Cost and FRP Area of California Highway Bridges (LS, Q (M)) with $R^2 = 0.898$ & $R^* = 0.806$

The regression equation obtained for accepted FRP cost against the FRP area for the bridges in California by LS regression method with an R^2 value of 0.991 is given by:

$$\text{Accepted FRP Cost (LS-\$)} = 11,609 + 60.12 * \text{FRP Area (ft}^2\text{)} \quad (15)$$

The regression equation obtained by Quantile (M) regression method for California bridges with R^* value of 0.926 is given by:

$$\text{Accepted FRP Cost (Q (M)-\$)} = 9,322 + 62.20 * \text{FRP Area (ft}^2\text{)} \quad (16)$$

Figure 3.4 shows the scatter plot of accepted FRP cost against FRP area of California bridges by the two analysis methods (LS, Q (M)) without the bridges - 08-

472304 & 08-0G4804. The high slope value of 62.20 for Quantile (M) and 60.12 for Least Square is due to higher FRP cost associated with the repairs. The slope difference (absolute) between the CERs of Figure 3.3 and Figure 3.4 is 0.90 (Q (M)) and 8.41 (LS) which indicates the robustness of Q (M) regression with respect to outliers.

The results from Table 3.6 show that Least Square regression method has the least SSE value and Quantile (M) regression method has the least MAD and MAPE values. The Quantile (M) regression method reduced the MAPE value from 10.01 to 4.25 than that of Least Square regression method. The major difference between the two methods is the better prediction of the Quantile method for the smallest area.

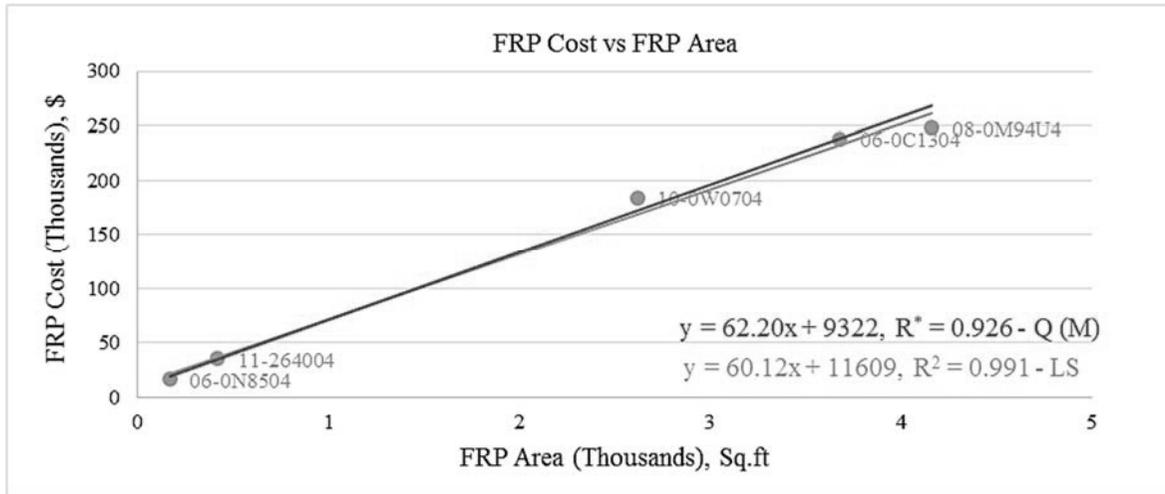


Figure 3.4: Scatter Plot of Accepted FRP Cost and FRP Area of California Highway Bridges (LS, Q (M)) with $R^2 = 0.991$ & $R^* = 0.926$

Table 3.6: Residual Analysis Table for California Highway Bridges

Area, ft ²	Accepted FRP Cost, \$	Predicted FRP Cost, \$		SSE (X-Xp) ²	
		Least Square	Quantile (Median)	Least Square	Quantile (Median)
170	16,763	21,830	17,898	25,676,719	1,287,365
420	35,449	36,861	35,449	1,992,953	0
2,620	183,974	169,129	172,302	220,370,462	136,228,091
3,677	238,053	232,678	238,054	28,889,830	1
4,163	248,157	261,897	268,286	188,797,658	405,183,847
Total	722,396	722,396	731,989	465,727,621	542,699,304
Area, ft ²	MAD		Average	MAPE, %	
	Least Square	Quantile (Median)		Least Square	Quantile (Median)
170	5,067	1,135		30.23	6.77
420	1,412	0		3.98	0
2,620	14,845	11,672		8.07	6.34
3,677	5,375	1		2.26	0
4,163	13,740	20,129		5.54	8.11
Average	8,088	6,587	Average	10.01	4.25

A relationship between the accepted FRP cost and the FRP area to predict the unit cost (\$/ft²) values, can be established. The predicted unit costs obtained from Equations 15 & 16 are given by:

$$\text{Unit Cost (LS-}\$/\text{ft}^2) = 60.12 + 11,609/\text{FRP Area}(\text{ft}^2) \quad (17)$$

$$\text{Unit Cost (Q (M)-}\$/\text{ft}^2) = 62.20 + 9,322/\text{FRP Area}(\text{ft}^2) \quad (18)$$

Table 3.7 shows the predicted FRP unit cost (\$/ft²) values obtained by the two methods (LS & Q (M)) of

analysis for individual bridges. The actual FRP unit costs versus predicted FRP unit costs have good agreement overall, but the predicted equations are approximate only. There is the assumption that the start-up costs, which is indicated by the intercept of the curve, is constant for all project sizes. The fit is better for the California equations as the intercept values are small than those for the Oregon equations. When the bridges are grouped into two ranges as done in Table 3.8, the agreement is much better for the unit costs.

Table 3.7: Summary of FRP Repair Unit Cost (\$/ft²) Obtained for Respective Areas of California Highway Bridges

Area, ft ²	Actual Cost, \$	Actual FRP Unit Cost, \$/ft ²	Pred. Unit Cost, \$/ft ² (LS)	Pred. Unit Cost, \$/ft ² (Q (M))
170	16,763	98.61	128.41	105.28
420	35,449	84.40	87.76	84.40
2,620	183,974	70.22	64.55	65.76
3,677	238,053	64.74	63.28	64.74
4,163	248,157	59.61	62.91	64.45

Table 3.8 shows for California bridges the FRP unit cost decreased as the area increased and the average FRP unit cost is approximately \$64/ft² for areas greater than 2,000 ft².

Table 3.8: Summary Average FRP Unit Cost (\$/ft²) Obtained for Respective Area Range of California Bridges

FRP Area Range, ft ²	Average* FRP Unit Cost, \$/ft ² (LS)	Average* FRP Unit Cost, \$/ft ² (Q (M))	Average* FRP Unit Cost, \$/ft ² (Actual)
<2000	99.47	90.42	88.49
>2000	63.45	64.88	64.07

*The area of the respective bridges was considered as weight in calculating the average FRP unit cost

4. Conclusions

There is a need for an effective cost estimation methodology in the transportation industry especially in the case of construction and rehabilitation of bridges. A brief introduction of Least Square regression and Quantile (M) regression was presented. The data of Oregon and California highway bridges was used to develop linear models for cost estimation of FRP wraps. The linear models obtained were used to compare the properties of the two methods based on graphical analysis and three test statistics (Mean Absolute Deviation (MAD), Mean Absolute Percentage Error (MAPE) & Sum of Square of Errors (SSE)). After the comparisons, the Quantile (M) regression had better properties in two aspects:

- Quantile (M) regression was found to have lower MAD and MAPE values, thus indicating the median improved the proximity of the CER to the data (Tables 3.2 & 3.6).
- Graphical analysis of results showed Quantile (M) regression to be robust to outliers. One or more outliers far away from the rest of the data had minimal impact on the predicted values. Quantile (M) regression did not favor high-cost values to the detriment of low-cost values as illustrated in Figures (3.1, 3.2, 3.3 & 3.4). The

slope difference (absolute) obtained between the CERs of Figures (3.1 & 3.2) of Oregon bridges is 0 (Quantile (M)) and 11.84 (Least Square) and for California bridges (i.e. between CERs of Figures (3.3 & 3.4)) is 8.41 (LS) and 0.90 (Q (M)) respectively. The low slope difference value for CERs of Quantile (M) regression validates the robustness of the method.

Based on the observations made in this study Quantile (Median) regression would be the preferred method of analysis when the data is large and scattered (Figure 3.1). When the mean and median are closer the differences become small between the two methods as indicated by Figure 3.4.

The analysis of the Oregon and California data was focused on developing relationships between FRP cost and FRP area by using Least Square and Quantile (M) analysis methods to predict the unit cost of FRP repair work. The high coefficient of determination values (0.884 (LS) & 0.773(Q) for Oregon, and 0.991(LS) & 0.926 (Q) for California) indicate FRP area to have a significant influence on the FRP cost. The FRP units cost curves are better predictors when the intercepts of

the linear FRP cost curves are small. The coefficient of determination values of Quantile (M) regression are lower than that of Least Square regression since the metric R^2 minimizes the sum of square of errors, which

is an inherent property of Least Square regression. The effect of outliers is much less with the Quantile (M) regression than with the Least Square regression.

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Appendix 1: Identifying The Outliers in Oregon Highway Bridges Data

Year	Bridge No.	FRP Area, ft ²	Accepted Total FRP Cost, \$	Predicted FRP Cost (ALL Points), \$	Deviation 1	Deviation 2	Relative Variation %	R ² (With Data Point Eliminated) 0.278
2007	01786A	3,144	159,343	105,282	54,062	57,829	6.5	0.281
2008	07601B	712	29,062	68,332	-39,270	-43,855	-7.91	0.251
2008	08381S	1,280	50,299	76,962	-26,662	-29,094	-4.19	0.265
2008	7458*	1,955	190,020	87,217	102,803	137,545	-59.93	0.352
2008	08383N	6,103	167,665	150,238	17,427	21,641	7.27	0.23
2009	8843*	2,345	250,480	93,142	157,337	168,017	18.42	0.449
2009	03173A*	9,204	124,069	197,351	-73,282	-159,297	-148.37	0.429
2010	7530	500	30,557	65,111	-34,555	-39,010	-7.69	0.248
2010	7532	500	30,557	65,111	-34,555	-39,010	-7.69	0.248
2010	7534	500	30,557	65,111	-34,555	-39,010	-7.69	0.248
2010	06945A	840	32,739	70,277	-37,538	-41,671	-7.13	0.254
2010	04981A	1,900	60,022	86,381	-26,360	-28,310	-3.36	0.273
2010	02233A	1,900	76,391	86,381	-9,990	-10,730	-1.27	0.275
2010	04979A	2,600	72,026	97,017	-24,991	-26,659	-2.88	0.28
2010	02236A	3,200	87,304	106,132	-18,828	-20,156	-2.29	0.282
2010	7392	5,800	174,609	145,634	28,975	35,016	10.43	0.227

*Outliers