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CUSP FORMS AND NUMBER OF REPRESENTATIONS OF POSITIVE INTEGERS BY DIRECT SUM OF BINARY QUADRATIC FORMS

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Abstract-*In this study, we calculated all reduced primitive binary quadratic forms which are* $F_1 = x_1^2 + x_1x_2 + 8x_2^2$, $\Phi_1 = 2x_1^2 + x_1x_2 + 4x_2^2$, $\Phi_1' = 2x_1^2 - x_1x_2 + 4x_2^2$. *We find the theta series* Θ_Q, *Eisenstein part of* Θ_Q *and the generalized theta series which are cusp forms by computing some spherical functions of second order with respect to Q . We obtain a basis of the subspace of* $S_4(\Gamma_0(31))$. *Explicit formulas are obtained for the number of representations of positive integers by all direct sum of three quadratic forms* $F_1 = x_1^2 + x_1x_2 + 8x_2^2$, $\Phi_1 = 2x_1^2 + x_1x_2 + 4x_2^2$, $\Phi_1' = 2x_1^2 - x_1x_2 + 4x_2^2$.

Keywords: *Positive Definite Quadratic Forms, Spherical Functions, Theta Series, Cusp Forms, Eisenstein Series*

1.INTRODUCTION

Modular forms have played an significant role in the mathematics of the 19th and 20th centuries, mostly in the theory of elliptic functions and quadratic forms. Quadratic forms occupy a central place in number theory, linear algebra, group theory, differential geometry, differential topology, Lie theory, coding theory and cryptology.
In this study, we focus on how to find a formula which solve problem of representation numbers of quadratic forms with discriminant -31. All calculations have been done by Maple.
Here, we will follow the method described in $[1,2,6]$ to determine the number of representations of some direct sum of quadratic forms of discriminant -31.
Let Δ be a negative integer such that

$$\Delta = \begin{cases} 4d & if \ \ d \equiv 2,3 \bmod 4 \\ d & if \ \ d \equiv 1 \bmod 4 \end{cases}$$

where *d* is square-free integer. It is called fundamental discriminant. Let $r(n;Q)$ denote the number of representations of *n* by *Q*.

Let $r(n;Q)$ denote the number of representations of *n* by *Q*. It is known that there exists a ont-to-one correspondence between $SL(2,\mathbb{Z})$ equivalence classes of positive definite binary quadratic forms

$$Q = ax^2 + bxy + cy^2$$

with integral coefficients of fundamental discriminant Δ and ideal classes of imaginary quadratic field $Q(\sqrt{d})$. In this correspondence, the number $r(n;Q)$ of representations of integer *n* by *Q*
$Q = n$
is equal to the number *w* of roots of 1 in $Q(\sqrt{d})$ times the number of ideals in the corresponding ideal class of norm *n*. Let

$$\Theta_Q(q) = \sum_{(x,y\in\mathbb{Z}\times\mathbb{Z})} q^{Q(x,y)} = \sum_{n=0}^{\infty} r(n;Q)\, q^n$$

be the theta function associated to positive definite quadratic form *Q*.

In this formulas Φ_1 can be replaced by its

It is known that it is a modular form of weight 1 with Dirichlet character

$$\chi(a) = \left(\frac{\Delta}{a}\right)$$

expressed by Kronecker symbol. In fact it is Legendre symbol if *a* is an odd prime.
There exist 3 inequivalent classes of binary quadratic forms of discriminant -31 whose reduced primitive binary quadratic forms are

$$F_1 = x_1^2 + x_1x_2 + 8x_2^2$$

$$\Phi_1 = 2x_1^2 + x_1x_2 + 4x_2^2$$

$$\Phi_1' = 2x_1^2 - x_1x_2 + 4x_2^2$$

Here, F_1 is the identity element. Φ_1' is the inverse of Φ_1.
Since -31 is prime number then there is only one genus, i.e., the principal genus.
F_k, Φ_k denote the *k* direct sum of F_1, Φ_1 respectively for $k \geq 1$. These binary quadratic forms form a group whose order is 3 such that

$$\Phi_1, \Phi_1^2 = \Phi_1', \Phi_1^3 = F_1$$

In this paper, formulas for $r(n;Q)$ are derived for any positive integer associated tot he following quadratic forms

$$Q = F_4, \Phi_4, F_1 \oplus \Phi_3, F_2 \oplus \Phi_2, F_3 \oplus \Phi_1.$$

inverse Φ_1'.

2.POSITIVE DEFINITE FORMS

Let $Q = ax^2 + bxy + cy^2$. A binary quadratic form is primitive if the integer *a*, *b* and *c* are relatively prime. Moreover, if $\Delta = b^2 - 4ac < 0$ and $a > 0$ then $Q(x,y)$ is positive definite. $M_k(\Gamma_0(N), \chi_\Delta)$ denotes the space of modular forms on $\Gamma_0(N)$ of weight *k*, with character χ_Δ. $S_{k+2}(\Gamma_0(N), \chi_\Delta)$ denotes the space of all cusp forms of weight *k*, with character χ_Δ. Definition 1 *Let Q be a positive definite quadratic form of 2k variables*

$$Q = \sum_{1 \le i \le j \le 2k}^{2k} b_{ij} x_i x_j, b_{ij} \in \mathbb{Z}$$

and the matrix A defined by

$$a_{ii} = 2b_{ii}, a_{ji} = a_{ij} = b_{ij} \;\; for \;\; i < j$$

Let D be the determinant of the matrix A and A_{ij} *the cofactors of A for* $1 \le i,j \le 2k$. *If* $\delta = gcd\left(\frac{A_{ii}}{2}, A_{ij} \; for \; 1 \le i,j \le 2k\right)$, *then* $N := \frac{D}{\delta}$ *is the smallest positive integer, called the level of Q, for which* NA^{-1} *is again an even integral matrix like A.* $\Delta = (-1)^k D$ *is called the discriminant of the form Q.*
Theorem 1 *Let* $Q: \mathbb{Z}^{2k} \to \mathbb{Z}$ *be a positive definite integer valued form of 2k variables of level N and discriminant* Δ. *Then*

1. *The theta function*

$$\Theta_Q(q) = \sum_{(n_1,n_2,\dots,n_k) \in \mathbb{Z}\times\mathbb{Z}\times\dots\times\mathbb{Z}} q^{Q(n_1,n_2,\dots,n_k)} = 1 + \sum_{n=1}^{\infty} r(n;Q)\, q^n, q = e^{2\pi i z} \;\; (*)$$

is a modular form on $\Gamma_0(N)$ *of weight k and character* χ_Δ, *i.e.,* $\Theta_Q \in M_k(\Gamma_0(N), \chi_\Delta)$, *where* $\chi_\Delta(d) := \left(\frac{\Delta}{d}\right)$, $d \in (\mathbb{Z}/N\mathbb{Z})^\times$, $\left(\frac{\Delta}{d}\right)$ *is the Kronecker character.*

2. *The homogeneous quadratic polynomials in 2k variables* $\varphi_{ij} = x_i x_j - \frac{1}{2k}\frac{A_{ij}}{D} 2Q, 1 \le i,j \le 2k$ *are spherical functions of second order with respect to Q. (**)*

3. *The theta series* $\Theta_{Q,\varphi_{ij}}(q) = \sum_{n=1}^{\infty}\left(\sum_{Q=n} \varphi_{ij}\right) q^n$ *is a cusp form in* $S_{k+2}(\Gamma_0(N), \chi_\Delta)$. (***)

4. *If two quadratic forms* Q_1, Q_2 *have the same level N and the characters* $\chi_1(d), \chi_2(d)$ *respectively, then the direct sum* $Q_1 \oplus Q_2$ *of the quadratic forms has the same level N and the character* $\chi_1(d).\chi_2(d)$.

Now, let's look at the positive definite quadratic forms of discriminant -31.

1- For the quadratic form $F_1 = x_1^2 + x_1x_2 + 8x_2^2$,

$$2F_1 = 2x_1^2 + 2x_1x_2 + 16x_2^2 = (x_1, x_2)\begin{pmatrix} 2 & 1 \\ 1 & 16 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

the determinant of the matrix and cofactors are $D = 31, A_{11} = 16, A_{12} = A_{21} = -1, A_{22} = 2$. So $\delta = 1, N = D = 31$ and the discriminant is $\Delta = (-1)^{2/2}31 = -31$. The character of F_1 is the Kronecker Symbol $\chi(d) = \left(\frac{-31}{d}\right)$.

2. For the quadratic form $\Phi_1 = 2x_1^2 + x_1x_2 + 4x_2^2$,

$$2\Phi_1 = 4x_1^2 + 2x_1x_2 + 8x_2^2 = (x_1, x_2)\begin{pmatrix} 4 & 1 \\ 1 & 8 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

the determinant of the matrix and cofactors are $D = 31, A_{11} = 8, A_{12} = A_{21} = -1, A_{22} = 4$. So $\delta = 1, N = D = 31$ and the discriminant is $\Delta = (-1)^{2/2}31 = -31$. The character of Φ_1 is the Kronecker Symbol $\chi(d) = \left(\frac{-31}{d}\right)$.

Consequently, F_1, Φ_1 are quadratic forms whose theta series are in $M_1\left(\Gamma_0(31), \left(\frac{-31}{d}\right)\right)$. Hence $F_2, \Phi_2, F_1 \oplus \Phi_1$ are quadratic forms whose theta series are in $M_2(\Gamma_0(31))$. Obviously there are only two inequivalent cusps $i\infty$ and 0 for $\Gamma_0(31)$.

Theorem 2 *Let Q be a positive definite form of 2k variables, $k = 4, 6, 8, \ldots$, whose theta series Θ_Q is in $M_k(\Gamma_0(p))$, p prime, then the Eisenstein part of Θ_Q is*

$$E(q; Q) = 1 + \sum_{n=1}^{\infty}(\alpha\sigma_{k-1}(n)q^n + \beta\sigma_{k-1}(n)q^{pn})$$

Where

$$\alpha = \frac{i^k}{\rho_k}\frac{p^{k/2} - i^k}{p^k - 1}$$

$$\beta = \frac{1}{\rho_k}\frac{p^k - i^k p^{k/2}}{p^k - 1}$$

$$\rho_k = (-1)^{k/2}\frac{(k-1)!}{(2\pi)^k}\zeta(k)$$

Corollary 1 *Let Q be a positive definite quadratic form of 8 variables whose theta series Θ_Q is in $M_4(\Gamma_0(31))$, then the Eisenstein part of Θ_Q is*

$$E(q; Q) = 1 + \sum_{n=1}^{\infty}(\alpha\sigma_3(n)q^n + \beta\sigma_3(n)q^{31n})$$

Where

$$\rho_4 = \frac{3!}{(2\pi)^4}\zeta(4) = \frac{1}{240}$$

$$\alpha = 240\,\frac{31^2 - 1}{31^4 - 1} = \frac{120}{481}$$

$$\beta = 240\,\frac{31^4 - 31^2}{31^4 - 1} = \frac{115320}{481}$$

3.SELECTION OF SPHERICAL FUNCTIONS

In order to find the generalized theta series corresponding to spherical functions, we will determine the sphericalfunctions of second order with respect to *Q*, see[3,9].

1.For the quadratic form

$$2F_2 = 2x_1^2 + 2x_1x_2 + 16x_2^2 + 2x_3^2 + 2x_3x_4 + 16x_4^2$$

$$= (x_1, x_2, x_3, x_4)\begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 16 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 16 \end{pmatrix},$$

the determinant $D = 31^2,\ A_{11} = 16.31$.

$$\varphi_{11} = x_1x_1 - \frac{1}{4}\frac{A_{11}}{D}2F_2 = x_1^2 - \frac{8}{31}F_2$$

which will be spherical function of second order with respect to F_2.

2.For the quadratic form

$$2\Phi_2 = 4x_1^2 + 2x_1x_2 + 8x_2^2 + 4x_3^2 + 2x_3x_4 + 8x_4^2$$

$$= (x_1, x_2, x_3, x_4)\begin{pmatrix} 4 & 1 & 0 & 0 \\ 1 & 8 & 0 & 0 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 1 & 8 \end{pmatrix},$$

the determinant $D = 31^2,\ A_{11} = 8.31, A_{12} = -31$

$$\varphi_{11} = x_1x_1 - \frac{1}{4}\frac{A_{11}}{D}2\Phi_2 = x_1^2 - \frac{4}{31}\Phi_2$$
$$\varphi_{12} = x_1x_2 - \frac{1}{4}\frac{A_{12}}{D}2\Phi_2 = x_1x_2 + \frac{1}{62}\Phi_2$$

which will be spherical functions of second order with respect to Φ_2.

3.For the quadratic form

$$2(F_1 \oplus \Phi_1) = 2x_1^2 + 2x_1x_2 + 16x_2^2 + 4x_3^2 + 2x_3x_4 + 8x_4^2$$

$$= (x_1, x_2, x_3, x_4)\begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 16 & 0 & 0 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 1 & 8 \end{pmatrix},$$

the determinant
$D = 31^2$, $A_{11} = 16.31, A_{12} = -31, A_{33} = 8.31$
.

$$\varphi_{11} = x_1x_1 - \frac{1}{4}\frac{A_{11}}{D}2(F_1 \oplus \Phi_1) = x_1^2 - \frac{8}{31}(F_1 \oplus \Phi_1)$$

$$\varphi_{12} = x_1x_2 - \frac{1}{4}\frac{A_{12}}{D}2(F_1 \oplus \Phi_1) = x_1x_2 + \frac{1}{62}(F_1 \oplus \Phi_1)$$

$$\varphi_{33} = x_3x_3 - \frac{1}{4}\frac{A_{33}}{D}2(F_1 \oplus \Phi_1) = x_3^2 - \frac{4}{31}(F_1 \oplus \Phi_1)$$

which will be spherical functions of second order with respect to $(F_1 \oplus \Phi_1)$.

Now, we will construct a basis of a subspace $S_4(\Gamma_0(31))$ of dimension 6. The general information about the modular forms $M_k(\Gamma_0(N), \chi)$ of weight *k* of the group $\Gamma_0(N)$ with Dirichlet character χ and the cusp forms $S_k(\Gamma_0(N), \chi)$ of weight *k* of the group $\Gamma_0(N)$ with Dirichlet character χ are given in details in [5,3,4,8].

Theorem 3 *The set of the following generalized 6 generalized theta series is a basis of the subpace of $S_4(\Gamma_0(31))$ spanned by all generalized theta series of the form (**) induced by spherical functions of the form (***).*

$$\Theta_{F_2,\varphi_{11}} = \frac{1}{31}\sum_{n=1}^{\infty}\left(\sum_{F_2=n} 31x_1^2 - 8F_2\right)$$

$$\Theta_{\Phi_2,\varphi_{11}} = \frac{1}{31}\sum_{n=1}^{\infty}\left(\sum_{F_2=n} 31x_1^2 - 4\Phi_2\right)$$

$$\Theta_{\Phi_2,\varphi_{12}} = \frac{1}{62}\sum_{n=1}^{\infty}\left(\sum_{F_2=n} 62x_1x_2 + \Phi_2\right)$$

$$\Theta_{(F_1\oplus\Phi_1),\varphi_{11}} = \frac{1}{31}\sum_{n=1}^{\infty}\left(\sum_{F_2=n} 31x_1^2 - 8(F_1 \oplus \Phi_1)\right)$$

$$\Theta_{(F_1\oplus\Phi_1),\varphi_{12}} = \frac{1}{62}\sum_{n=1}^{\infty}\left(\sum_{F_2=n} 62x_1x_2 + (F_1 \oplus \Phi_1)\right)$$

$$\Theta_{(F_1\oplus\Phi_1),\varphi_{33}} = \frac{1}{31}\sum_{n=1}^{\infty}\left(\sum_{F_2=n} 31x_3^2 - 4(F_1 \oplus \Phi_1)\right)$$

Proof. The series are cusp forms because of Theorem 1.
Therefore, the generalized theta series associated to spherical functions can be calculated as follows:

$$\Theta_{F_2,\varphi_{11}} = \frac{1}{31}\sum_{n=1}^{\infty}\left(\sum_{F_2=n} 31x_1^2 - 8F_2\right)$$

$$= \frac{1}{31}(30q + 60q^2 + 120q^4 + 300q^5 + \cdots)$$

$$\Theta_{\Phi_2,\varphi_{11}} = \frac{1}{31}\sum_{n=1}^{\infty}\left(\sum_{F_2=n} 31x_1^2 - 4\Phi_2\right)$$

$$= \frac{1}{31}(30q^2 - 4q^4 - 18q^5 - 68q^6 - 26q^7 + \cdots)$$

$$\Theta_{\Phi_2,\varphi_{12}} = \frac{1}{62}\sum_{n=1}^{\infty}\left(\sum_{F_2=n} 62x_1x_2 + \Phi_2\right)$$

$$= \frac{1}{31}(4q^2 + 16q^4 - 52q^5 + 24q^6 - 20q^7 + \cdots)$$

$$\Theta_{(F_1 \oplus \Phi_1),\varphi_{11}} = \frac{1}{31}\sum_{n=1}^{\infty}\left(\sum_{F_2=n} 31x_1^2 - 8\,(F_1 \oplus \Phi_1)\right)$$

$$= \frac{1}{31}(46q - 32q^2 + 28q^3 + 120q^4 - 116q^5 + 236q^6 - 112q^7 + \cdots)$$

$$\Theta_{(F_1 \oplus \Phi_1),\varphi_{12}} = \frac{1}{62}\sum_{n=1}^{\infty}\left(\sum_{F_2=n} 62x_1x_2 + (F_1 \oplus \Phi_1)\right)$$

$$= \frac{1}{31}(q + 2q^2 + 6q^3 + 8q^4 + 15q^5 + 24q^6 + 7q^7 + \cdots)$$

$$\Theta_{(F_1 \oplus \Phi_1),\varphi_{33}} = \frac{1}{31}\sum_{n=1}^{\infty}\left(\sum_{F_2=n} 31x_3^2 - 4\,(F_1 \oplus \Phi_1)\right)$$

$$= \frac{1}{31}(-8q + 46q^2 + 76q^3 - 64q^4 - 58q^5 + 56q^6 + 6q^7 + \cdots)$$

4.CONCLUSION

According to (*) we can obtain $\Theta_{F_1} = 1 + 2q + 2q^4 + \cdots$ and $\Theta_{\Phi_1} = 1 + 2q^2 + 2q^4 + 2q^5 + 2q^7 + \cdots$.
Then we can obtain theta series of quadratic forms $F_4, \Phi_4, F_1 \oplus \Phi_3, F_2 \oplus \Phi_2, F_3 \oplus \Phi_1$ by direct sum of Θ_{F_1} and Θ_{Φ_1}. By subtracting any one of these theta series by Eisenstein series, we get a linear combination of the generalized theta series.

$$r(n;Q) = \Theta_Q(q) - E(q;Q) = c_1\Theta_{F_2,\varphi_{11}}(q) + c_2\Theta_{\Phi_2,\varphi_{11}}(q) + c_3\Theta_{\Phi_2,\varphi_{12}}(q) + c_4\Theta_{F_1 \oplus \Phi_1,\varphi_{11}}(q) + c_5\Theta_{F_1 \oplus \Phi_1,\varphi_{12}}(q) + c_6\Theta_{F_1 \oplus \Phi_1,\varphi_{33}}(q)$$

.

By equating the coefficients of q^n in both sides for $n = 1,2,3,4,5,6,7$, we can find out $c_1, c_2, c_3, c_4, c_5, c_6$.
From these identities, we get the formulas for $r(n;F_4), r(n;\Phi_4), r(n;F_1 \oplus \Phi_3), r(n;F_2 \oplus \Phi_2), r(n;F_3 \oplus \Phi_1)$.
(See [6])

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