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ON THE FUZZY CONDITIONAL SEQUENCE LOCAL ENTROPY FUNCTION

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Abstract- *In this work,we first recall some basic properties of the fuzzy conditional sequence entropy function without going into details.After that,we define the fuzzy conditional sequence local entropy function.Lastly,we prove some important results relating to this function.*

Keywords: *Fuzzy dynamical system, fuzzy complete system,conditional entropy function, fuzzy sequence entropy function,fuzzy conditional sequence entropy function,fuzzy local entropy function,fuzzy conditional sequence local entropy function.*

1. INTRODUCTION

Hulse (3) and Zhang (13) first, introduced the concept of conditional sequence entropy function and investigated some properties of this function in the non-fuzzy sense.
The author defined the fuzzy conditional sequence entropy function and stated some important properties of this function in (8).In another its work (10), the author has recently introduced the notion of the fuzzy conditional local entropy function and investigated some important results relating to this function.
It is the purpose of this paper to define the fuzzy conditional sequence local entropy function and show some fundamental results of this function.

2. FUZZY CONDITIONAL SEQUENCE ENTROPY FUNCTION

Let (X,**A**,m,T) be a dynamical system.Where (X,**A**,m) is a Lebesgue space and T : (X,**A**,m) → (X,**A**,m) is an invertible measure-preserving transformation.We shall also refer to (X,T) instead of (X,**A**,m,T) for convenience.For details ,see,(2) and (11).

2.1 Definition. A dynamical system (Y,**B**,m_1,S) is called a factor of the dynamical system (X,**A**,μ,T) if there exists a measure-preserving function φ : X → Y such that for all x∈X, φ(T(x)) = S(φ (x)).Equivalently,we say that (X,**A**,m,T) is an extension of (Y,**B**,m_1,S).

2.2 Definition

Let (Y,S) be a factor of the dynamical system (X,T) and **P** ={p_1,p_2,…,p_n} be a finite measurable partition of (X,T).Then the quantity

$$H_m(P/B)=\sum_{i=1}^{n}\int z(E_m(\chi_{p_i}/B))dm=-\sum_{i=1}^{n}m(p_i/B)\log(m(p_i/B))$$

is called the conditional entropy function of a finite measurable partition **P** with respect to the σ-algebra **B**.Where $E_m(\chi_{p_i}/B)$ is a conditional expectation of the characteristic function χ_{p_i} defined by $\chi_{p_i}(x)=\begin{cases}1 & if\ x\in p_i\\ 0 & if\ x\notin p_i\end{cases}$ for i = 1,…,n ,m(p_i / B) is conditional measure of p_i with respect to the σ-algebra **B** defined by

$$m(p_i/B)=\frac{m(p_i\cap B)}{m(B)}$$

for i = 1,…,n ,with m(B) > 0 and z : [0,∞) → R defined by

$$z(x)=\begin{cases}-x\log x & if\quad x>0\\ 0 & if\quad x=0\end{cases}$$

is a non-negative,continuous and strictly concave function.In this work,all logarithms will be to the natural base ″ e ″.For more properties of the conditional expectation, see,(1) (2) and (11).

2.3 Proposition

Let (Y,S) be a factor of the dynamical system (X,T) and **P** and **Q** be two finite measurable partitions of (X,T) with H_m (**P/B**) < ∞ and H_m (**Q/B**) < ∞ .Then,

(**i**) H_m (**P / B**) ≥ 0.

(**ii**) If **P** ⊂ **B**,then H_m (**P / B**) = 0.

(**iii**) If **B** = { X,ϕ },then H_m (**P/ B**) = H_m(**P**). Where H_m(**P**) is an entropy function of the finite measurable partition **P** defined by

$H_m(P) = \sum_{i=1}^{n} z(m(p_i))$ .For details,see,(1) and (11).

(**ıv**) If **P** ⊂ **Q**,then H_m (**P/ B**) ≤ H_m (**Q/B**).

(**v**) H_m (**P/ B**) ≤ H_m (**P**).

(**vı**) H_m (**P** ∨ **Q/ B**) = H_m (**P/ B**) + H_m (**Q/ P** ∨ **B**).

(**vıı**) H_m(**P** ∨ **Q/ B**) ≤ H_m (**P/ B**) + H_m (**Q/ B**).For measurable partitions with finite conditional entropy equality holds if and only if **P** and **Q** are independent

i.e. m(**P** ∩ **Q**) = m(**P**) . m(**Q**).

(**vııı**) If T is a measure-preserving transformation,then H_m (T^{-1}**P**/T^{-1}**B**) = H_m (**P/ B**).

Proof. See,(1) and (11).

2.4 Definition

Following Zadeh (12),a pair (X,**F**) is called a fuzzy set.Where X is an arbitrary non-empty set and A : X → [0,1] is a membership function.That is,a fuzzy set is characterized by a membership function A from X to the closed unit interval I = [0,1].Thus,

we can identify a fuzzy set with its membership function A.In this connection,A(x) is interpreted as the degree of membership of a point x∈X.The family of all fuzzy sub sets is called a fuzzy class and will be denoted by **F.**This family F is called a fuzzy class**.** For details,see,(5) and (12).

Let **(X,F**,μ) be a fuzzy probability measure space.Where X is a fuzzy set, **F** is a fuzzy σ-algebra **F** defined on the a fuzzy set X and μ is a fuzzy probability measure defined on the fuzzy measurable space **(X,F)**.The elements of **F** are fuzzy measurable events.For more properties of the fuzzy probability measure space **(X,F**,μ) ,see ,(5) and (6).

2.5 Definition

Let **(X,F**,μ) be a fuzzy probability measure space.

(i) The collection **P** = { $A_1,\ldots,A_n$}of fuzzy sub sets is called disjoint,if $(\bigvee_{i=1}^{j} A_i) \wedge A_{j+1} = \phi$ for each j = 1,2,…,n-1.

(**ii**) A collection **P =** { $A_1,\ldots,A_n$ } of disjoint fuzzy sub sets is called a finite fuzzy partition if and only if $X = \bigvee_{i=1}^{n} A_i$.

(**iii**) A collection **P =** { $A_1,\ldots,A_n$ }with A_i ∈ **F** for i = 1,2,…,n-1 is called a complete system of fuzzy events if and only if **P** is a fuzzy partition of X.

(**ıv**) Let **P** and **Q** be two fuzzy complete systems.Then,**P** and **Q** are independent

if and only if μ(**P . Q**) = μ (**P**) . μ (**Q**)**.** Where **P . Q** is a fuzzy product of **P** and **Q** defined by (**P . Q**) (x) = **P**(x) . **Q**(x) for all x ∈ X.For details, see,(5),(6) and (12).

2.6 Definition

Let (X,**F**,μ) be a fuzzy probability measure space.The mapping

T : (X,**F,** μ) → (X,**F,** μ) is called a σ-homomorphism if it satisfies the following properties;

(**i**) $T(\overline{A}) = \overline{T(A)}$ for every A ∈ **F**.

(**ii**) $T(\bigvee_n A_n) = \bigvee_n T(A_n)$ for any fuzzy sequence $(A_n)_{n\in N}$ ⊂ **F**.

(**iii**) μ(TA) = μ(A) for each A ∈ **F.**

The quadruple (X,**F**,μ,T) is called a fuzzy dynamical system.One will write briefly (X,T) instead of (X,**F,**μ,T) for convenience.For more properties of the fuzzy dynamical system,see,(4) and (6)

2.7 Theorem

Suppose that the dynamical system (Y,S) is a factor of the dynamical system (X,T).Let **P** be a finite fuzzy complete system of the fuzzy dynamical system (X,T) with H_μ(**P/ F_1**) < ∞ and let

$D = (t_n)_{n\geq 1}$ be a sequence of integers with $t_1 = 0$.Then $\lim_{n\to\infty} \sup \frac{1}{n} H_\mu(\bigvee_{i=1}^{n} T^{t_i} P / F_1)$ exists.

Proof. See ,Theorem 3.4 of (8).

2.8 Definition

Let (Y,S) be a factor of the dynamical system (X,T) and **P** be a finite fuzzy complete system of (X,T) with H_μ (**P**/ **F**$_1$) < ∞ and let $D = (t_n)_{n\geq 1}$ be a sequence of integers with $t_1 = 0$.Then the limit function

$$h_{\mu,f,D}(T, P/F_1) = \lim_{n\to\infty} \sup \frac{1}{n} H_\mu(\bigvee_{i=1}^{n} T^{t_i} P / F_1)$$

is called the fuzzy conditional sequence entropy function of finite fuzzy complete system **P** with respect to the fuzzy σ-algebra **F**$_1$.

2.9 Proposition

Suppose that the dynamical system (Y,S) is a factor of the dynamical system (X,T).Let **P** , **Q** be two finite fuzzy complete systems of (X,T) with H_μ (**P**/ **F**$_1$) < ∞ and H_μ (**Q**/ **F**$_1$) < ∞ and $D = (t_n)_{n\geq 1}$ be a sequence of integers with $t_1 = 0$.Then,

(i) $h_{\mu,f,D}$ (T,**P**/ **F**$_1$) ≤ $h_{\mu,f,D}$ (T,**P**).Where $h_{\mu,f,D}(T;P) = \lim_{n\to\infty}\sup \frac{1}{n} H_\mu(\bigvee_{i=1}^{n} T^{t_i} P)$ is called the fuzzy sequence entropy function of T with respect to the finite complete system **P** with H_μ (**P**) < ∞.(For details See,(7)).

(ii) If **P** ⊂ **Q**,then $h_{\mu,f,D}$ (T,**P**/ **F**$_1$) ≤ $h_{\mu,f,D}$ (T,**Q**/ **F**$_1$).
(iii) $h_{\mu,f,D}$ (T,**P** ∨ **Q**/ **F**$_1$) ≤ $h_{\mu,f,D}$ (T,**P**/ **F**$_1$) + $h_{\mu,f,D}$ (T,**Q**/ **F**$_1$).For finite fuzzy complete systems with finite fuzzy conditional entropies equality holds if and only if **P** and **Q** are independent.

Proof. See ,Proposition 3.7 of (8).

2.10 Proposition

Suppose that the fuzzy dynamical system (Y,S) is a factor of the fuzzy dynamical system (X,T).Let **P** be a finite complete system of (X,T) with H_μ (**P**/ **F**$_1$) < ∞ and let $D = (t_n)_{n\geq 1}$ be a sequence of integers with $t_1 = 0$.Then,
(i) $h_{\mu,f,D}$ (T,**P** / **F**$_1$) ≥ 0.

(ii) If **P** ⊂ **F**$_1$,then $h_{\mu,f,D}$ (T,**P** / **F**$_1$) = 0.

(iii) If **F**$_1$ = { X,ϕ },then $h_{\mu,f,D}$ (T,**P**/ **F**$_1$) = $h_{\mu,f,D}$(T,**P**).

(ιv) If T is a fuzzy probability measure-preserving σ-homomorphism ,then

$h_{\mu,f,D}$ (T,T^{-1}**P**/T^{-1} **F**$_1$) = $h_{\mu,f,D}$ (T,**P**/ **F**$_1$).

Proof . **(i)**,**(ii)** and **(iii)** follow easily from Definition 2.2 and Proposition 2.3 (i),(ii) and (iii). **(ιv)** If T is a fuzzy probability measure-preserving σ-homomorphism, we have from the Proposition 2.3 (vııı)

H_μ (T^{-1}**P**/T^{-1} **F**$_1$) = H_μ (**P**/ **F**$_1$) (2.1).Therefore, we can also write the following equality;

$$H_\mu(\bigvee_{i=1}^{n} T^{t_i}(T^{-1}P/T^{-1}F_1)) = H_\mu(\bigvee_{i=1}^{n} T^{t_i} P/F_1)$$
(2.2).

Dividing the equality (2.2) by n > 0 and taking the superior limit for n → ∞ , we obtain the result from the Theorem 2.7 and Definition 2.8;
$h_{\mu,f,D}$ (T,T^{-1}**P**/T^{-1} **F**$_1$) = $h_{\mu,f,D}$ (T,**P**/ **F**$_1$) (2.3)

2.11 Definition

Let (Y,S) be a factor of fuzzy dynamical system (X,T) and let **P** be a fuzzy finite complete system of (X,T) with H_μ (**P**/ **F**$_1$)<∞.We cosider $D = (t_n)_{n\geq 1}$ a sequence of integers with $t_1 = 0$.Then the quantity
$h_{\mu,f,D}$ (T/ **F**$_1$) = {$h_{\mu,f,D}$ (T,**P**/ **F**$_1$) : **P** is a finite fuzzy complete system of X with H_μ(**P**/ **F**$_1$) < ∞} is called the fuzzy conditional sequence entropy function of the fuzzy dynamical system (X,T).Where the supremum is taken over all finite fuzzy complete systems with the finite fuzzy conditional entropies.

2.12 Proposition

(i) $h_{\mu,f,D}$ (T/ **F**$_1$) ≥ 0.

(ii) If **P** is a σ-sub algebra of **F**$_1$,then $h_{\mu,f,D}$ (T/ **F**$_1$) = 0.

(iii) If **F**$_1$ = { X,ϕ },then $h_{\mu,f,D}$ (T/ **F**$_1$) = $h_{\mu,f,D}$ (T).

Where $h_{\mu,f,D}$ (T) = { $h_{\mu,f,D}$ (T,**P**) : **P** is a finite fuzzy complete system of X with H_μ (**P**)<∞ }
is a fuzzy sequence entropy function of the fuzzy dynamical system (X,T).For details See, (7)
(ιv) $h_{\mu,f,D}$ (T/ **F**$_1$) ≤ $h_{\mu,f,D}$ (T).

Proof . **(i)**,**(ii)** **(iii)** and **(ιv)** follow easily from Proposition 2.9 (i),(ii) , Proposition 2.10 (i),(ii) and (iii) and Definition 2.11.

2.13 Proposition. Let (Y,S) be a factor of the fuzzy dynamical system (X,T) and

$D = (t_n)_{n\geq 1}$ be a sequence of integers with $t_1 =$ 0.Then $h_{\mu_1,f,D}(S/F_1) \leq h_{\mu,f.D}(T/F_1)$.

Proof. Let φ be a fuzzy probability measure

preserving function.If **Q** is finite fuzzy complete system of (Y,S) with $H_{\mu_1}(Q/F_1) < \infty$,then φ^{-1}**Q** is a finite fuzzy complete system of (X,T) with H_μ (φ^{-1}**Q/ F**$_1$) < ∞.

Then,it is easy to see that

$$\varphi^{-1}(\overset{n}{\underset{i=1}{V}} S^{t_i}Q) = \overset{n}{\underset{i=1}{V}}(\varphi^{t_i}Q) \quad (2.4).$$

Therefore.we can also write the following equality,

$$H_\mu(\varphi^{-1}(\overset{n}{\underset{i=1}{V}} S^{t_i}Q)/F_1) = H_\mu(\overset{n}{\underset{i=1}{V}} T^{t_i}(\varphi^{-1}Q/F_1)$$
(2.5).

Dividing the equality 2.5 by n>0 and taking the superior limit for n → ∞,we obtain the following equalities from the Theorem 2.7 and Definition 2.8;

$h_{\mu,f,D}$ (φ^{-1}S,**Q/ F**$_1$) = $h_{\mu,f,D}$ (T,φ^{-1}**Q/ F**$_1$) (2.6) and also

$$h_{\varphi o \mu,f,D}(S,Q/F_1) = h_{\mu_1,f,D}(S,Q/F_1) = h_{\mu,f,D}(T,\varphi^{-1}Q/F_1)$$
(2.7).

Hence,we have from the Definition 2.11,

$$h_{\mu_1,f,D}(S/F_1) = \sup_Q\{h_{\mu_1,f,D}(S,Q/F_1) : Q \textit{ is a finite fuzzy complete system of } Y \textit{ with } H_{\mu_1}(Q/F_1) < \infty\}$$
by the Equality 2.7

$= \sup_{\varphi^{-1}Q}$ {$h_{\mu,f,D}$ (T,φ^{-1}**Q/ F**$_1$):φ^{-1}**Q** is a finite fuzzy complete system of X with H_μ (φ^{-1}**Q/ F**$_1$)<∞}
by the Proposition 2.9 (iii)

$\leq \sup_P$ { $h_{\mu,f,D}$ (T,**P/ F**$_1$) : **P** is a finite fuzzy complete system of X with H_μ (**P/ F**$_1$)<∞ }(2.8) yazılır.

Therefore the result follows from the Definition 2.
$h_{\mu_1,f,D}$ (S/ **F**$_1$) ≤ $h_{\mu,f,D}$ (T/ **F**$_1$) (2.9).

3. FUZZY CONDITIONAL SEQUENCE LOCAL ENTROPY FUNCTION

3.1 Definition. Let (Y,S) be a factor of the fuzzy dynamical system (X,T) and let **P** be a finite fuzzy complete system of (X,T) with H_μ (**P/ F**$_1$)<∞ and $D = (t_n)_{n\geq 1}$ be a sequence of integers with $t_1 = 0$. Then,the quantity $L_{\mu,f,D}$ (T/ **F**$_1$) = $h_{\mu,f,D}$(T/ **F**$_1$) - $h_{\mu,f,D}$ (T,**P/ F**$_1$)

is called the fuzzy conditional sequence local entropy function.

3.2 Proposition. (**i**) $L_{\mu,f,D}$ (T/ **F**$_1$) ≥ 0.

(**ii**) If **P** ⊂ **F**$_1$,then $L_{\mu,f,D}$(T/ **F**$_1$) = 0.

(**iii**) If **F**$_1$ = {X,ϕ},then $L_{\mu,f,D}$ (T/ **F**$_1$) = $L_{\mu,f,D}$ (T).Where $L_{\mu,f,D}$ (T) = $h_{\mu,f,D}$ (T) - $h_{\mu,f,D}$ (T,**P**)

is a fuzzy sequence local entropy function.For details,see,(9).

(**ıv**) $L_{\mu,f,D}$ (T/ **F**$_1$) ≤ $L_{\mu,f,D}$ (T).

Proof .(i) This follows from the Proposition 2.10 (**i**),Proposition 2.12 (**i**) and Definition 3.1.

(**ii**) If **P** ⊂ **F**$_1$,then we write the following equalities from the Proposition 2.10(**ii**) Proposition 2.12 (**ii**) and Definition 3.1,

$h_{\mu,f,D}$ (T,**P/ F**$_1$) = 0 (3.1) and $h_{\mu,f,D}$ (T/ **F**$_1$) = 0 (3.2).

Hence,we obtain the result from the Defnition 3.1,

$L_{\mu,f,D}$ (T/ **F**$_1$) = 0 (3.3).

(**iii**) If **F**$_1$ = { X,ϕ },we write the following equalities from the Proposition 2.10 (**iii**) and

Proposition 2.12 (**iii**),

$h_{\mu,f,D}$ (T,**P/ F**$_1$) = $h_{\mu,f,D}$ (T,**P**) (3.4) and $h_{\mu,f,D}$ (T/ **F**$_1$) = $h_{\mu,f,D}$(T) (3.5).

Therefore, we can also write the following equality;

$h_{\mu,f,D}$ (T/ **F**$_1$) - $h_{\mu,f,D}$ (T,**P/ F**$_1$) = $h_{\mu,f,D}$ (T) - $h_{\mu,f,D}$ (T,**P**) (3.6).

Hence, the result follows from the Definition 3.1,

$L_{\mu,f,D}$ (T/ $\mathbf{F}_1$) = $L_{\mu,f,D}$ (T) (3.7).

(**ıv**) Let **P** be a finite fuzzy complete system of (X,T) with H_μ (**P**/ $\mathbf{F}_1$)<∞ .Then,we have from the Proposition 2.9 (**i**) and Proposition 2.12 (**ıv**),

$h_{\mu,f,D}$ (T,**P**/ $\mathbf{F}_1$) ≤ $h_{\mu,f,D}$ (T,P) (3.8) and $h_{\mu,f,D}$ (T/ $\mathbf{F}_1$) ≤ $h_{\mu,f,D}$ (T) (3.9)

Therefore, we write the following equality;

$h_{\mu,f,D}$ (T/ $\mathbf{F}_1$) - $h_{\mu,f,D}$ (T,**P**/ $\mathbf{F}_1$) ≤ $h_{\mu,f,D}$ (T) - $h_{\mu,f,D}$ (T,**P**) (3.10).

Hence,we obtain the result from the Defnition 3.1, $L_{\mu,f,D}$ (T/ $\mathbf{F}_1$) ≤ $L_{\mu,f,D}$ (T) (3.11).

3.3 Proposition. We consider that (Y,S) is a factor of fuzzy dynamical system (X,T).Let **P** be a a finite fuzzy complete system of (X,T) with H_μ(**P**/ $\mathbf{F}_1$) < ∞ and **Q** be a finite fuzzy complete system of (Y,S) with $H_{\mu_1}(Q/F_1) < \infty$ and D = $(t_n)_{n\geq 1}$ be a sequence of integers with

t_1 = 0.Then,

$$L_{\mu_1,f,D}(S/F_1) \leq L_{\mu,f,D}(T/F_1) + h_{\mu,f,D}(T,P/F_1) - h_{\mu_1,f,D}(S,Q/F_1)$$

Proof. If (Y,S) is a factor of fuzzy dynamical system (X,T),then we have the following inequality from the Proposition 2.13,

$$h_{\mu_1,f,D}(S/F_1) \leq h_{\mu,f,D}(T/F_1) \qquad (3.12)$$

Let **P** be a finite fuzzy complete system of (X,T) with H_μ (**P**/$\mathbf{F}_1$) < ∞.

Since $h_{\mu,f,D}$(T,**P**/$\mathbf{F}_1$) ≥ 0,from the Proposition 2.10 (i),we can write the following inequality, $h_{\mu_1,f,D}(S/F_1) - h_{\mu,f,D}(T,P/F_1) \leq h_{\mu,f,D}(T/F_1) - h_{\mu,f,D}(T,P/F_1)$ (3.13)

Therefore,we obtain from the Definition 3.1, $h_{\mu_1,f,D}(S/F_1) \leq L_{\mu,f,D}(T/F_1) + h_{\mu,f,D}(T,P/F_1)$ (3.14)

Let **Q** be a finite fuzzy complete system of (Y,S) with $H_{\mu_1}(Q/F_1) < \infty$.

As $h_{\mu_1,f,D}(S,Q/F_1) \geq 0$,from the Propsition 2.10 (i),we can also write the following inequlity;

$$h_{\mu,f,D_1}(S/F_1) - h_{\mu_1,f,D}(S,Q/F_1) \leq L_{\mu,f,D}(T/F_1) + h_{\mu,f,D}(T,P/F_1) - h_{\mu_1,f,D}(S,Q/F_1)$$
(3.15)

Hence, the result follows from the Definition 3.1,

$$L_{\mu,f,D}(S/F_1) \leq L_{\mu,f,D}(T/F_1) + h_{\mu,f,D}(T,P/F_1) - h_{\mu_1,f,D}(S,Q/F_1)$$
(3.16)

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